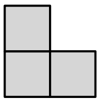
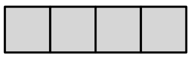
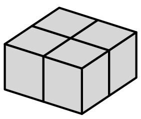
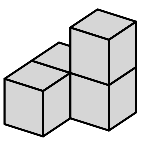
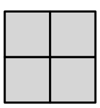
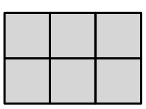
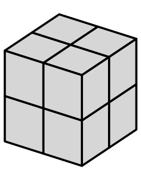
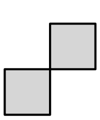
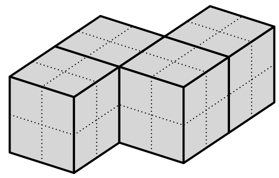
Week 7

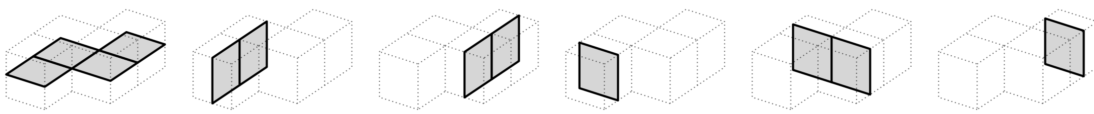
Here are a bunch of cubical complexes (i.e. "shapes made out of squares and cubes attached together along their faces"). We'll ask some questions about them on the next few pages.

Cubical complex	# vertices, edges, squares, 3D cubes	
	$8, 10, 3, 0$	
	$10, 13, 4, 0$	
	$18, 33, 20, 4$	
	$20, 36, 21, 4$	
	$9, 12, 4, 0$	
	$12, 17, 6, 0$	
	$27, 54, 36, 8$	
	$7, 8, 2, 0$	

Question 1. Here is a cubical complex.



The dotted lines show the “hyperplanes” of the cubical complex: a hyperplane is a slice through the complex that cuts through the middle of some of the cubes. Here are the six hyperplanes of this cubical complex on their own:



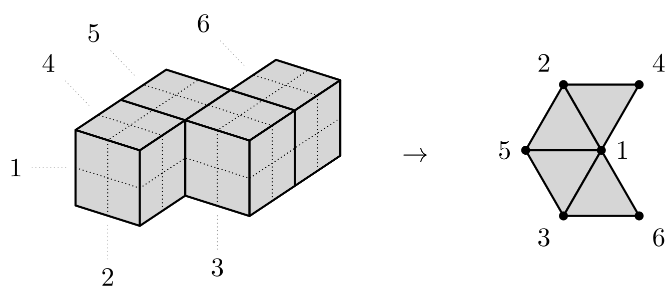
Look at the cubical complexes on the previous page, and draw their hyperplanes. How many hyperplanes does each complex have?

(Work on this question until you get the hang of what hyperplanes are, then move on.)

Question 2. Every cubical complex has something called a “crossing complex”. The crossing complex is a *simplicial* complex (i.e. a shape built out of simplices — line segments, triangles, tetrahedrons, etc. — attached along their sides). Here’s how to build it:

- The crossing complex has one **vertex** for each **hyperplane** of the cubical complex.
- A set of vertices of the crossing complex has a **simplex** between them if the corresponding hyperplanes in the cubical complex **all cross each other** somewhere.

Here is the crossing complex for our example cubical complex:

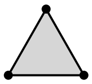


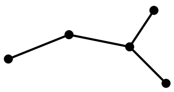
For example, there is a triangle with vertices 1, 2 and 4 because the hyperplanes 1, 2 and 4 all intersect in the middle of the left-most cube.

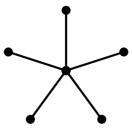
Draw the crossing complex for each cubical complex on the previous page.

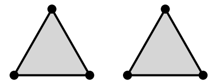
Question 3. Can you design a cubical complex which has each of these simplicial complexes as its crossing complex?

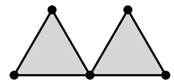
(Note: Some of these are challenging, feel free to skip them.)

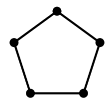
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

Question 4. Do you notice any relationships between the cubical complexes and their crossing complexes? (We'll spend the rest of the worksheet exploring this question!)

Question 5. For the cubical complexes on page 1 and their crossing complexes:

(a) How many shapes is each complex made of?

(b) What dimensions are the shapes?

(c) *(Challenge.)* Can you explain why your observations in this question happen?

Question 6. For a simplicial complex, there is something called its "f-polynomial" — it's a polynomial with this formula:

$$1 + (\#\text{vertices}) \cdot t + (\#\text{edges}) \cdot t^2 + (\#\text{triangles}) \cdot t^3 + (\#\text{tetrahedrons}) \cdot t^4$$

(a) Calculate the f-polynomials for the crossing complexes.

(b) What do you get if you set $t = -1$?
(Hint: Ask someone who was here last week.)

(c) Try plugging in $t = s + 1$, and simplifying. What do you notice?

Question 7. *(Challenge.)* How many holes does each crossing complex have?

What modification can you make to the cubical complexes so that they have the same holes as their crossing complexes?