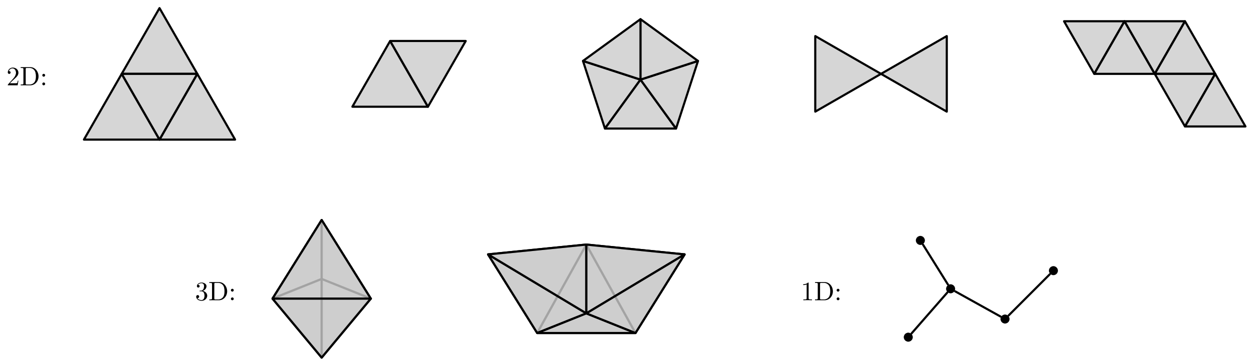
Week 6

Last week, we studied simplices and cubes. This week, we'll look at *simplicial and cubical complexes*, i.e. shapes built from multiple simplices or cubes!

Question 1. Here are some simplicial complexes.



(a) For each of these simplicial complexes, count the numbers of vertices (corners), edges, triangles and tetrahedra.

For the 3D ones, remember to count the stuff in the inside of the shape!

(b) For each simplicial complex, calculate:

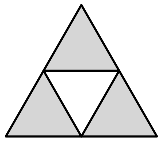
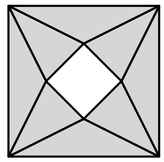
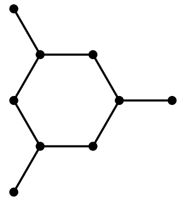
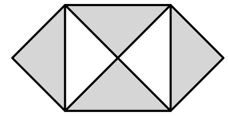
$$-1 + (\#\text{vertices}) - (\#\text{edges}) + (\#\text{triangles}) - (\#\text{tetrahedra})$$

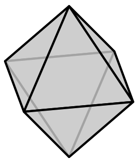
What do you notice?

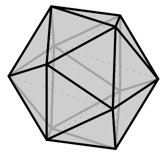
This number is called the "Euler characteristic" of the simplicial complex.

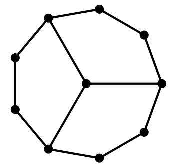
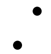
(c) Design your own simplicial complex, and calculate this formula again.

(d) Here are some more simplicial complexes.


(this shape is hollow)

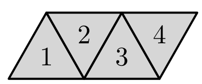

(this one's hollow too)

Calculate their Euler characteristics. What's different about these simplicial complexes?

Question 2. A simplicial complex is called "shellable" if you can build it one simplex at a time, always attaching each new simplex so that it touches the old simplices along some of its sides, and not along anything smaller.[1] If it's possible to do this, the order that you attach the simplices is called a "shelling order".

(a) Here's a simplicial complex:



Which of these orders are shelling orders?

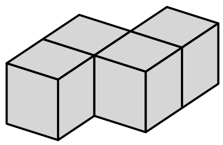
$$1, 2, 3, 4 \qquad 1, 3, 4, 2 \qquad 3, 2, 4, 1 \qquad 4, 1, 3, 2$$

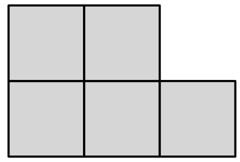
(b) Which of the simplicial complexes in the previous questions are shellable?

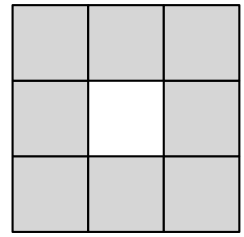
(c) What can you say about the numbers of vertices, edges, etc. for a shellable simplicial complex: how do these numbers change when each new simplex is attached? Along one side, two sides, three sides. . . ? What effect does this have on the Euler characteristic?

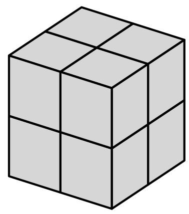
[1] The formal definition (for instructors' reference) says that $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ must be a pure, $(\dim F_i)$-dimensional subcomplex of F_i for all $i > 1$.

Question 3. Here are some cubical complexes:

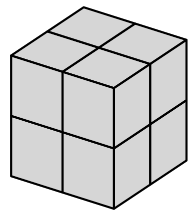








(filled)



(hollow)

(a) Calculate their Euler characteristics. For cubical complexes, the formula is

$$-1 + (\#\text{vertices}) - (\#\text{edges}) + (\#\text{squares}) - (\#\text{3D cubes})$$

Can you guess what the answers will be before calculating them. . . ?

(b) A cubical complex can be “shellable” too. The definition is the same: you have to attach each new cube so it touches the old cubes along its sides.

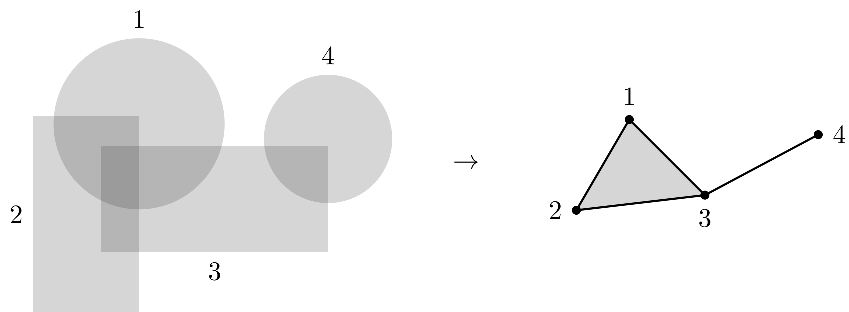
If you attach a new cube, how do the numbers of vertices, edges, etc. change, and how does this affect the Euler characteristic?

Question 4. Simplicial complexes are often useful ways of recording information about mathematical scenarios.

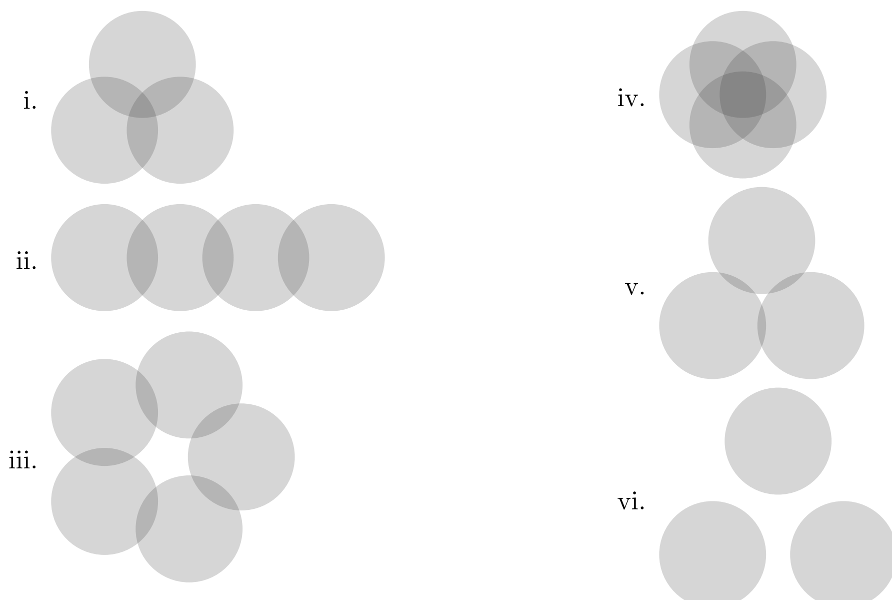
Suppose we have an arrangement of overlapping shapes. We can build a simplicial complex (called the "nerve" of the arrangment) that records which shapes overlap each other.

This simplicial complex has one vertex for each shape in the arrangement. A set of vertices has a simplex between them if the corresponding shapes all overlap at some point.

Here's an example:



(a) Do this for some more arrangements of shapes.



(b) What do you notice about the geometry?

Question 5. *(Challenge.)* We can explain the pattern with the Euler characteristic using shellings, but not every simplicial/cubical complex is shellable. Can you think of a way to explain the Euler characteristic pattern for complexes that aren't shellable?