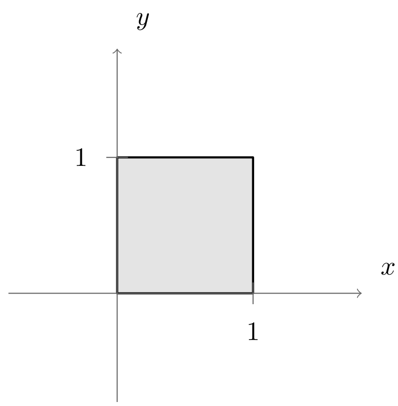
Week 5

This week we'll study some shapes: **cubes** and **simplices**!

Question 1. Let's start with squares. This is a square:



(This isn't the only way to position a square in the 2D plane, but it's the one we'll look at today. It's called the "standard unit square".)

(a) Here are the coordinates of some points:

$(0,1)$ $(0.2,0.7)$ $(2.3,0.4)$ $(-1,0)$ $(0,0.6)$ $(0.3,-0.4)$ $(0,0)$ $(0.3,1)$ $(0.5,0.4)$

Which of these points are:

i. . . . on the square?

ii. . . . on an edge of the square?

iii. . . . a corner of the square?

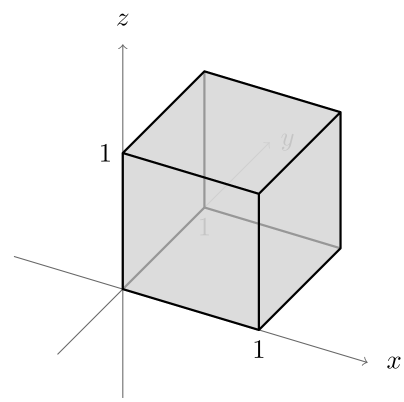
(b) Figure out a rule to decide whether a point (x, y) is in the square or not.

(c) Count some things:

i. How many corners does the square have?

ii. How many edges?

Question 2. This is a cube:



(Specifically, this is the “standard unit cube” in 3D space.)

Cubes are kind of like “3D versions of squares”. Let’s see if we can add some detail and justification to that statement.

(a) Here are the coordinates of some points:

$$(1,0,1) \quad (0,-1,1) \quad (0.6,0.3,0.8) \quad (0,0,0.2) \quad (0.3,1,2) \quad (-1,-1,-1)$$
$$(0,0,0) \quad (0,0.5,1) \quad (0.8,0.9,1) \quad (1,0,0) \quad (0.2,1,1)$$

Which of these points are:

i. . . . in the cube?

ii. . . . on one of the square sides of the cube?

iii. . . . on one of the edges of the cube?

iv. . . . a corner of the cube?

(b) Figure out a rule to decide whether a point (x, y, z) is in the cube or not.

(c) Count some things:

i. How many corners does the cube have?

ii. How many edges?

iii. How many square sides?

Question 3. We've looked at a 2D cube (i.e. a square) and a 3D cube (i.e. a cube). Can we extend this pattern to higher dimensions — what about a 4D cube?

(a) Points in 4D space are represented with 4 coordinates, just like how a 3D point has 3 coordinates and a 2D point has 2 coordinates. For example, $(3, 6, 5, 1)$ is a point in 4D.

Based on your answers from the last two pages, what rule do you think we should use to decide whether a 4D point (x, y, z, w) is in the standard unit 4D cube?

(b) Based on your rule, which of these points are in the 4D cube?

$(1, 1, 0, 0)$ $(0.2, 0.4, 0.7, 0.3)$ $(3, -1, 5, 2)$ $(0.3, 0.4, 19, 0.7)$ $(0, 0, 1, 0)$ $(1, 0, 1, 0.6)$
$(0.2, 1, 1, 0.9)$ $(0.3, 0.8, 1, 0.5)$ $(0.8, 0.8, 1, 0)$ $(0, 0, 0, 0)$ (0.3,

(c) Which of these points do you think are corners of the 4D cube?

Which ones do you think are on the "sides" of the 4D cube?

(d) How many corners do you think the 4D cube has?

How many edges? How many sides? (*Bonus:* What shape would you guess the sides are?)

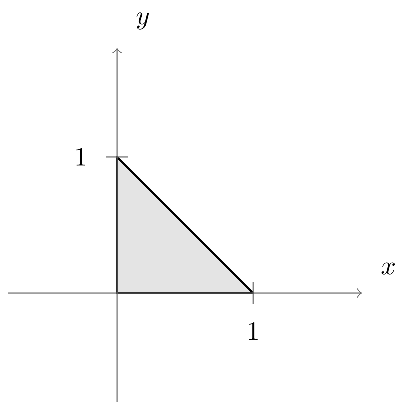
Question 4. Now, answer all the questions on this page for an n-dimensional cube, instead of a 4-dimensional cube!

What happens if $n = 1$? Do your answers make sense?

What about $n = 0$? Discuss.

So, that was cubes. Now, let's discuss simplices. ("Simplices" is the plural of "simplex": you can say "one simplex, two simplices".)

Question 5. Let's start in 2D again. Here's a triangle:



(a) Label the vertices (corners) of the triangle with A, B and C.
For each edge of the triangle, list the vertices that touch it.
Are there any pairs of vertices that don't have an edge between them?

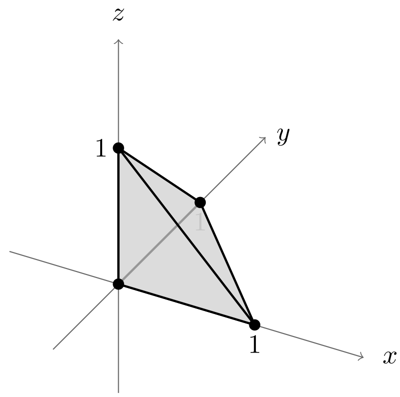
(b) How many vertices are there? How many edges?
What are the coordinates of the vertices?

(c) *(Challenge.)* Which of these points are in the triangle?

$$(0,1) \qquad (2,5) \qquad (0.1,0.1) \qquad (0.1,0.6) \qquad (0.8,0.9)$$
$$(0.8,0) \qquad (0.8,0.1) \qquad (0.8,0.2) \qquad (0.8,0.3)$$

What's the general rule?

Question 6. Now, let's step up to 3D. This is a tetrahedron:



(It's like a triangular-based pyramid, but a bit wonky.)

Let's ask the same questions again.

(a) Label the vertices (corners) of the tetrahedron with A, B, C and D.
For each edge of the tetrahedron, list the vertices that touch it. Do the same for each triangular side.
Are there any pairs or triples of vertices that don't have an edge or a triangle between them?

(b) How many vertices are there? How many edges, and how many sides?
What are the coordinates of the vertices?

(c) *(Challenge.)* Which of these points are in the tetrahedron?

$$(0,0,0) \quad (1,0,0) \quad (1,1,0) \quad (9,9,9) \quad (0,0,0.3) \quad (0.2,0.2,0)$$
$$(0.1,0.2,0.3) \quad (0.5,0.5,0.5) \quad (0.3,0.1,0.6)$$

What's the general rule?

Question 7. And finally, higher dimensions! The n-dimensional version of triangles and tetrahedrons is called a "simplex". How do you think it works?

Have a go at describing the vertices of an n-simplex. How many vertices are there? What are their coordinates?

How many edges do you think there will be? How many sides? (What shape do you think the sides will be?)

What does a simplex look like in 1D?

(Challenge.) What do you think the general rule should be to decide whether an n-dimensional point is inside the n-dimensional simplex?

Question 8. *(Bonus questions.)*

(a) The vertices, edges, sides etc. of a shape are collectively called "faces". What's the total number of faces for an n-dimensional cube, or an n-dimensional simplex?

(b) There is a slightly different way we could position a simplex in n-dimensional space: the rule is the same as the one from this worksheet, except with the additional requirement that the sum of the coordinates must equal 1. Draw what this rule looks like in 2D and 3D.

(c) There's a different 3D shape called a "cross polytope". It's defined by a single rule: $|x| + |y| + |z| \leq 1$. What does this shape look like? Count the vertices, edges, sides — what's the connection with a cube? Can you generalise this shape to other dimensions?

(d) You can build a square out of two triangles stuck together. Can you build a 3D cube out of 3D simplices? If you can, how many do you need?