Week 1

Question 1. Are the following numbers even or odd?

(a) 1234×567

(b) $45836 \times 7823 + 89273 \times 9231$

(c) 2^{84}

(d) 17^{429}

(e) $10^{10^{10}}$

(f) $23^{23^{23}}$

(g) $1^1 + 2^2 + 3^3 + \cdots + 9^9 + 10^{10}$

(h) $129!$

(i) The 100th Fibonacci number

(j) The 999th triangle number

Question 2. What is the last digit of each of the numbers from question 1?

Question 3. Which of these numbers are divisible by 3?

The three questions on the previous page can be answered in a similar way.

- The last digit of a number is the same as the remainder when you divide the number by 10.
- The remainder when you divide by 2 is 1 if the number is odd and 0 if it's even.
- A number is divisible by 3 whenever the remainder when you divide by 3 is 0.

If two numbers a and b have the same remainder when you divide them by q, then we say that "a and b are congruent modulo q", written "$a \equiv b \pmod{q}$". For example, 36 is congruent to 86 modulo 10 (written "$36 \equiv 86 \pmod{10}$"), since 36 and 86 both have the same remainder (namely 6) when you divide them by 10. Here are some more examples:

$$12 \equiv 5 \pmod{7}, \qquad 124 \equiv 54 \pmod{10},$$
$$38 \equiv 0 \pmod{2}, \qquad 100 \equiv 898 \pmod{3}.$$

(Warning: this is a little different than the "mod" operator you might have used in computer programming! Here, "mod" isn't an ***operation*** *you do to a number, it's more like a* ***context*** *you can check equivalence in.)*

Question 4. True or false:

(a) $843643538 \equiv 5345636 \pmod{10}$

(b) $843643538 \equiv 5345636 \pmod{2}$

(c) $843643538 \equiv 5345636 \pmod{5}$

(d) $843643538 \equiv 5345636 \pmod{3}$

(e) $843643538 \equiv 5345636 \pmod{9}$

(f) $843643538 \equiv 5345636 \pmod{57}$

Question 5. Prove these facts:

(a) If the remainder when you divide a by q is r, then $a = nq + r$ for some integer n.

(b) These two statements are equivalent:

$$a_1 \equiv a_2 \pmod{q},$$

and

$$a_1 - a_2 \text{ is divisible by } q.$$

(Hint: To prove two statements are equivalent, start by assuming the first is true and proving that the second follows from it — then assume the second is true, and prove that the first follows.)

Question 6. Suppose that $a_1 \equiv a_2 \pmod{q}$ and $b_1 \equiv b_2 \pmod{q}$. Are the following statements true or false?

(a) $a_1 + b_1 \equiv a_2 + b_2 \pmod{q}$. (For example, $10004 + 10003 \equiv 4 + 3 \pmod{10}$.)

(b) $a_1 \times b_1 \equiv a_2 \times b_2 \pmod{q}$

(c) $a_1 \div b_1 \equiv a_2 \div b_2 \pmod{q}$

(d) ${a_1}^{b_1} \equiv {a_2}^{b_2} \pmod{q}$

Question 7. Prove the true equations in the previous question. *(Hint: Use part (b) of question 5.)* Can you modify the false ones to make them true?

According to the previous page, if you want to find the remainder of $a + b$ when you divide by q, you don't actually need to add a and b: you can take the remainders of a and b modulo q first, then just add the remainders together. This is useful if a and b are very big numbers, for example.

Question 8. What does this have to do with questions 1 to 3?

Question 9. What is the remainder when you:

(a) ...divide $3256 + 1982$ by 10?

(b) ...divide $(703 + 356 + 79 + 3)$ by 7?

(c) ...divide $(7846217 + 439878142 + 87437632)$ by 100?

(d) ...divide $(3614 + 3614 + 3614 + 3614 + 3614)$ by 36?

(e) ...divide (2802×5) by 14?

(f) ...divide (6793×65) by 8?

(g) ...divide (197×197) by 196?

Question 10. Fill in the blanks:

(a) $1^7 \equiv$ ____ (mod 7)

(b) $2^7 \equiv$ ____ (mod 7)

(c) $3^7 \equiv$ ____ (mod 7)

(d) $4^{11} \equiv$ ____ (mod 11)

(e) $2^{13} \equiv$ ____ (mod 13)

(f) $8^5 \equiv$ ____ (mod 5)

What's the pattern?

This pattern doesn't work for all numbers — investigate when the pattern works and when it doesn't.