Week 9

This week, we're going to work with two important ideas:

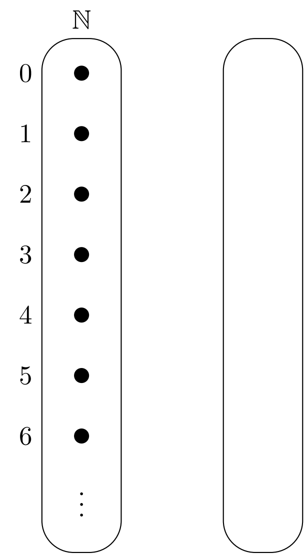
- If you can match up every thing in set A with a thing in set B so that every thing in each set is matched exactly once, then A and B have the same size. (In last week's terminology, this matching is a "bijective function".)
- If you can match up things in A with a subset of the things B so that everything in the subset is matched exactly once, then B is at least as big as A. (Last week we called this an "injective function".)

This week, we'll use these ideas to study the sizes of infinite sets!

Question 1. The "natural numbers", also called "$\mathbb{N}$", are the set of non-negative integers:

$$\text{natural numbers} = \{0, 1, 2, 3, \dots\} = \mathbb{N}$$

Let's look at some other infinite sets that are the same size as $\mathbb{N}$. Find a way to match up the natural numbers with each of the following sets:



(a) The non-positive numbers, $\{\dots, -3, -2, -1, 0\}$

(b) The natural numbers without 0

(c) The non-negative even numbers, $\{0, 2, 4, 8, \dots\}$

(d) The square numbers

(e) All integers (not just the non-negative ones)
(Hint: Split the natural numbers into odd and even ones, and split the "all integers" set into positive and negative ones.)

Note: According to (b), (c) and (d), an infinite set can be the same size as a subset of itself!

Question 2. Next, let's look at another set: $\mathbb{Q}$, the set of all fractions. Here are some things in $\mathbb{Q}$:

$$\frac{3}{5}, \qquad \frac{22}{7}, \qquad -\frac{1}{9999}, \qquad \frac{8723623}{598204}, \qquad -\frac{17}{6}, \qquad \frac{5}{1} = 5$$

Before we think about the size of $\mathbb{Q}$ as a set, let's think about how $\mathbb{Q}$ fits in the number line.

Explain why these facts are true:

(a) For any two fractions, we can always find another fraction between them.
(Hint: Consider their average.)

(b) In fact, for any two real numbers, we can always find a fraction between them.
(Hint: Every decimal number that eventually recurs is a fraction.)

(c) *(Challenge:)* Between any two real numbers, we can always find an *irrational* number (that is, a number that is NOT a fraction).

Question 3. How does this compare to the natural numbers? If I give you two natural numbers, can you always find another natural number between them?

Question 4. Before you look at the next page: how do you think thes sizes of $\mathbb{N}$ (natural numbers) and $\mathbb{Q}$ (fractions) compare? Do you think $\mathbb{Q}$ will be bigger, smaller or the same size?

Question 5. It turns out that $\mathbb{N}$ and $\mathbb{Q}$ are actually the same size! Let's figure out why.

(a) First: let's prove that $\mathbb{Q}$ is at least as big as $\mathbb{N}$. Describe a way to match up $\mathbb{N}$ with a subset of $\mathbb{Q}$.

(b) Next, let's prove that $\mathbb{N}$ is at least as big as $\mathbb{Q}$ by matching $\mathbb{Q}$ with a subset of $\mathbb{N}$. Pick your favourite fraction in $\mathbb{Q}$, and follow along with the steps below.

1. Write your fraction in the form "p/q".
2. Convert p and q to binary.
3. Replace the "/" symbol with a "2". If your fraction is negative, replace the "$-$" at the front with a "3".
4. Read off the result as a base 10 integer using the digits 0, 1, 2 and 3.

Does this procedure always work: does it always take a fraction in $\mathbb{Q}$ and produce a natural number in $\mathbb{N}$?

Can two different fractions ever be matched to the same natural number?

So, $\mathbb{Q}$ is at least as big as $\mathbb{N}$, and $\mathbb{N}$ is at least as big as $\mathbb{Q}$. Therefore, they're the same size!

Question 6. *(Challenge:)* Instead of matching each of $\mathbb{N}$ and $\mathbb{Q}$ with a subset of each other (i.e. two injective functions), can you describe a way to match $\mathbb{N}$ with all of $\mathbb{Q}$ (i.e. a bijective function)?

So far, all the infinite sets we've looked at have been the same size. Perhaps every possible infinite set is the same size. . . ?

Question 7. Next, let's think about the "real numbers", denoted "$\mathbb{R}$": this is the set of everything you can write as a decimal. For example:

$$\pi, \qquad -737.373737\cdots, \qquad 3+\sqrt{2}, \qquad 0.12345678910111213\cdots, \qquad \frac{5}{6}, \qquad -18$$

Before you read further: do you think $\mathbb{R}$ will be bigger than $\mathbb{N}$, or smaller, or the same size?

Question 8. Surprise — $\mathbb{R}$ is NOT the same size as $\mathbb{N}$!

Suppose we could match up the natural numbers with the real numbers so that every real number appears in the list somewhere. For example:

$$\begin{array}{ccrcccccccc}
0 & \leftrightarrow & 3 & . & 1 & 4 & 1 & 5 & 9 & 2 & \cdots \\
1 & \leftrightarrow & 116 & . & 5 & 3 & 9 & 0 & 2 & 2 & \cdots \\
2 & \leftrightarrow & 7 & . & 2 & 5 & 0 & 0 & 0 & 0 & \cdots \\
3 & \leftrightarrow & 11 & . & 0 & 9 & 0 & 5 & 3 & 6 & \cdots \\
4 & \leftrightarrow & 24 & . & 1 & 6 & 6 & 6 & 6 & 6 & \cdots \\
\vdots & & \vdots & & & & & & & &
\end{array}$$

I claim: no matter how we try to match up the numbers like this, we'll always miss out at least one real number. Let me show you how to write down a number that we've missed. (Give it a try yourselves!)

1. To the left of the decimal place, write something different to the stuff left of the decimal place in the 0th number in the matching.
2. For the first decimal place, choose a digit that's different to the first decimal place of the 1st number in the matching.
3. For the second decimal place, write a digit that's different to the second decimal place of the 2nd number in the matching.
4. And so on: for the nth decimal place, write a digit that's different to the nth decimal place of the nth number in the matching.

Why does this process produce a number that doesn't appear in the matching?

Question 9. In summary: the sizes of infinite sets are weird!

Discuss the following questions with your group.

(a) We saw that $\mathbb{N}$ and $\mathbb{Q}$ are the same size, but $\mathbb{R}$ is bigger. Did this surprise you? How do you feel about this? (Do you believe the math?)

(b) We know that there are at least two different sizes of infinity. Do you think there are more? How many different sizes of infinity do you think there are?

(c) The infinity of $\mathbb{R}$ is bigger than the infinity of $\mathbb{N}$. Do you think there's anything in between — could there be a set that's bigger than $\mathbb{N}$, but not as big as $\mathbb{R}$? How many different sizes of infinity do you think there are in the gap between $\mathbb{N}$ and $\mathbb{R}$?

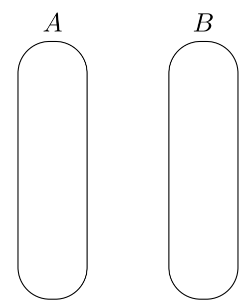
(d) Can you think of any other infinite sets which you're curious to know the size of?

Question 10. Remember from last week:

- A **surjective** function is a way to match each thing in A with something in B, where everything in B is matched **at least once**. This tells you that A is at least as big as B.
- An **injective** function is a way to match each thing in A with something in B so that everything in B is matched **at most once**. This tells you that B is at least as big as A.
- A **bijective** function is a way to match each thing in A with something in B, where every thing in B is matched **exactly once**. This tells you that A and B are the same size.

Let's make sure these three ways of comparing size actually fit together, and that "size" works the way we think it does, even for infinite sets! Explain the following facts:

(a) If there's a bijective function from A to B, then we can find a bijective function from B to A.



(b) If there's an injective function from A to B, then we can find a surjective function from B to A.

(c) If there's a surjective function from A to B, then we can find an injective function from B to A.

(d) If there is an injective function from A to B and another injective function from B to A, then we can find a bijective function from A to B.

(This one's tough.

Start with something in one of the sets. Instead of following the arrows forwards, what happens if you follow them backwards? Is it possible to get stuck? What possibilities are there for what can happen?

We want to build a bijective function using some of the arrows in the first function and some arrows (backwards) from the second function. Find every thing in A or B where it takes an ***even number*** *of steps before getting stuck when you follow the arrows backwards — for each of these things, take the arrow* ***pointing out*** *of it, and put that arrow in the bijective function. What should we do with the other things? Why does this make a bijective function?)*