Week 8

Question 1. Let's play a game of Which Is Bigger?

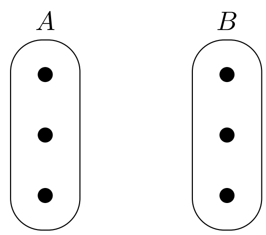
Each round, I'll name two collections of things. Your task is to decide which collection has more things in it! Warning — some answers might not be possible to figure out.

- Cars vs. Red cars
- Cars vs. Car manufacturing companies
- Cars that were made before 1990 vs. Cars that were made before 2010
- Cars owned by Taylor Swift vs. Cars owned by you
- Yellow cars vs. Purple cars
- Cars vs. Car wheels
- Cars vs. Car steering wheels
- Cars vs. Engines
- Cars vs. Sun roofs
- Cars vs. Cup holders
- Cars vs. Cars
- Cars vs. Cats

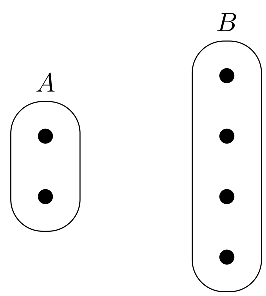
A *function* from a collection A to another collection B is a way to draw arrows from things in A to things in B, so that **every thing in A has exactly one arrow coming out of it**.

Question 2. Draw a function from the collection of dots on the left to the collection of dots on the right.

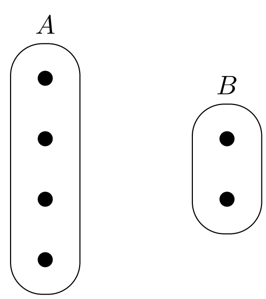
(a)



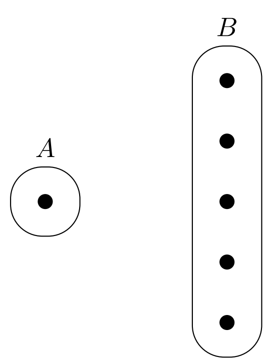
(b)



(c)

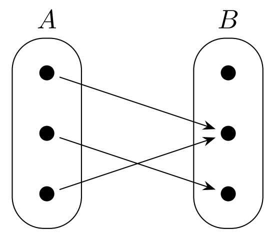


(d)

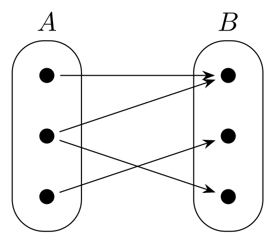


Question 3. Which of these pictures are valid functions?

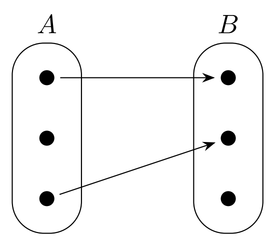
(a)



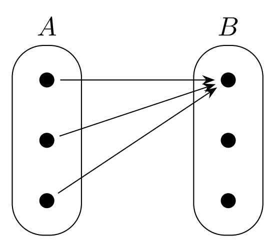
(b)



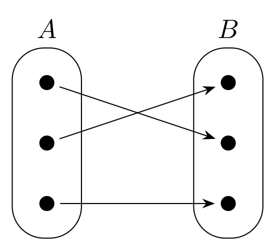
(c)



(d)



(e)



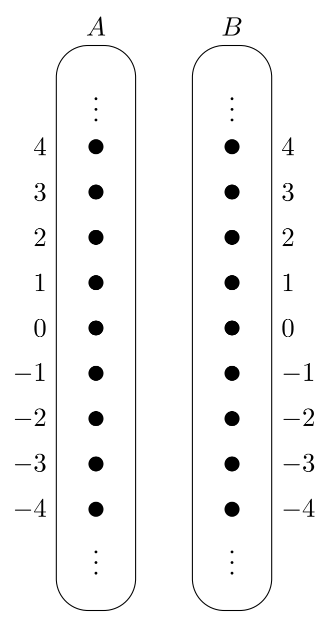
Question 4. A function from A to B is *surjective* if every thing in B has at least one arrow pointing to it.
Which of the functions on the previous page are surjective?
What does a surjective function tell you about the sizes of the two collections?

Question 5. A function from A to B is *injective* if every thing in B has at most one arrow pointing to it.
Which of the functions on the previous page are injective?
What does an injective function tell you about the sizes of the two collections?

Question 6. A function from A to B is *bijective* if it is both injective and surjective.
Which of the functions on the previous page are bijective?
What does a bijective function tell you about the sizes of the two collections?

Question 7. Draw functions from the collection of all integers to the collection of all integers, according to the following rules. Which of these functions are surjective, or injective, or bijective?

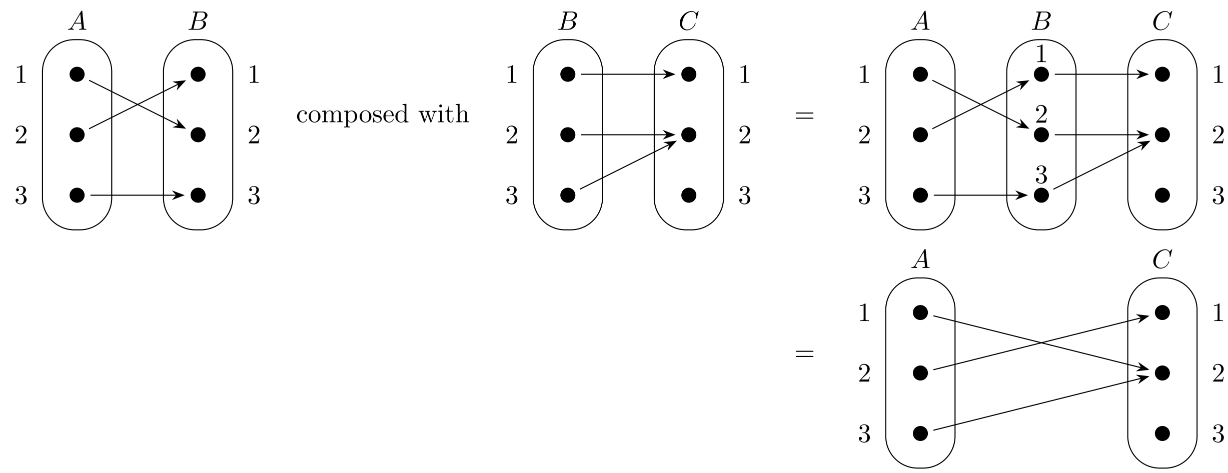
(a) Send n to $n + 1$.
(b) Send n to $2n$.
(c) Send n to n^2.
(d) Send n to $n \div 2$, rounded down.
(e) Send n to n.



Question 8. Go back to page 1, and try to find surjective, injective or bijective functions that explain the answers to the questions. (The functions might go from the first collection to the second, or maybe the other way around!)

Question 9. If I give you a function f from A to B, and a function g from B to C, you can *compose* those functions to get a new function from A to C: just write the functions next to each other, and follow the arrows!

For example:



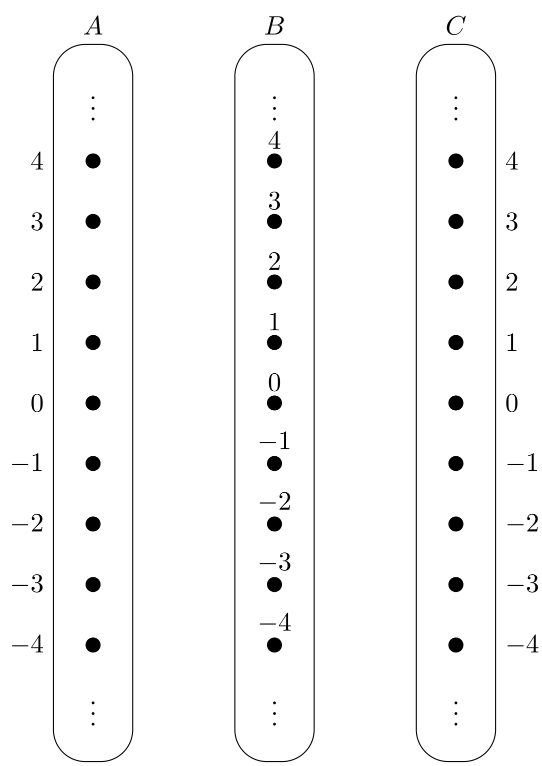
What do you get if you compose these functions from the collection of integers to itself:

(a) The $m \mapsto m^2$ function, then the $n \mapsto n+1$ function

(b) The $m \mapsto m+1$ function, then the $m \mapsto 2m$ function

(c) The $m \mapsto 2n$ function, then the $n \mapsto n \div 2$ rounded down function

(d) The $m \mapsto m \div 2$ rounded down function, then the $n \mapsto 2n$ function



Are each of these functions surjective, injective or bijective?

Question 10. Can you find functions f and g so that:

(a) f, g and their composition are all surjective?

(b) f, g and the composition are all injective?

(c) f and g are surjective, but their composition isn't?

(d) f is injective, g is surjective, and the composition is neither

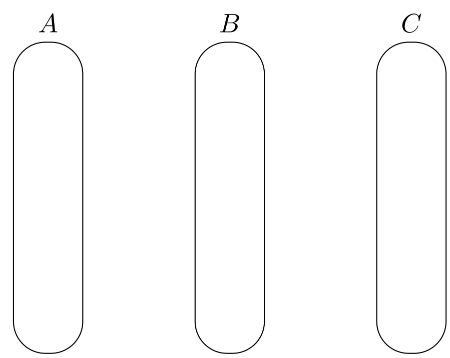
(e) f is surjective, g is injective, and the composition is neither

(f) f is surjective, g is injective, and the composition is both

(g) The composition is bijective, but f and g are both not bijective

(h) Between f, g and their composition, exactly two are bijective

Here are some empty blobs for you to draw on. You can choose how many dots they should have.



What other combinations are possible?

Question 11. Suppose A and B have the same size. Can you find a function from A to B that is injective, but not surjective? Or a function that's surjective, but not injective?

Question 12. Suppose f is a function from A to B. When can you find a function g from B to A so that the composition of f and g takes every thing in A to itself — what can you say about A, B, f and g?

What if the composition of g and f also sends every thing in B to itself?

Question 13. Suppose collection A contains a things, and B contains b things. How many functions are there from A to B?

Does your formula make sense if $a = 0$, or $b = 0$? Or if $a = \infty$ or $b = \infty$?

Suppose $b = 2$ — can you think of anything else that's counted by the same formula? What's the connection?

How many surjective functions are there, or injective functions, or bijective functions?