Week 5

Let's look at some applications of binomial coefficients. Remember:

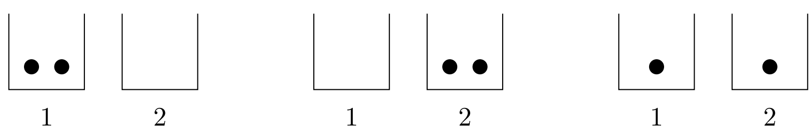
$$\binom{n}{k} = \#\text{ ways to choose } k \text{ things out of a set of } n \text{ things} = \frac{n!}{k!(n-k)!}$$

Question 1. I have $2n$ Christmas lights, each of which can be set to Red or Green. How many ways can I light them up so that there are exactly n lights of each colour?

n	1	2	3	4	5	6
$2n$	2	4	6	8	10	12
# ways						

Question 2. I have m identical marbles and b numbered boxes to put them in. How many ways can I put the marbles in the boxes? I can put multiple marbles in each box, and I can leave some boxes empty.

For example, if I have 2 marbles and 2 boxes, I can put both marbles in box 1, or both in box 2, or one marble in each. (In this last case, since the marbles are all identical, it doesn't matter which one I put in box 1 or 2 — count them as the same arrangement.)



Number of arrangements:

		m marbles					
		1	2	3	4	5	6
b boxes	1						
	2						
	3						
	4						
	5						
	6						

Bonus challenge problem: can you explain why these numbers are related to binomial coefficients?

Question 3. Compute the powers of 11. What do you notice?

n	11^n
0	
1	
2	
3	
4	
5	

Can you explain this pattern?

(Hint: $11 \times$ *something* $= 10 \times$ *something* $+$ *something.)*

Try the powers of 101 as well.

Question 4. *(Optional problem, for groups where everyone knows some algebra)*

Here are some algebra rules you can use to multiply out complicated expressions:

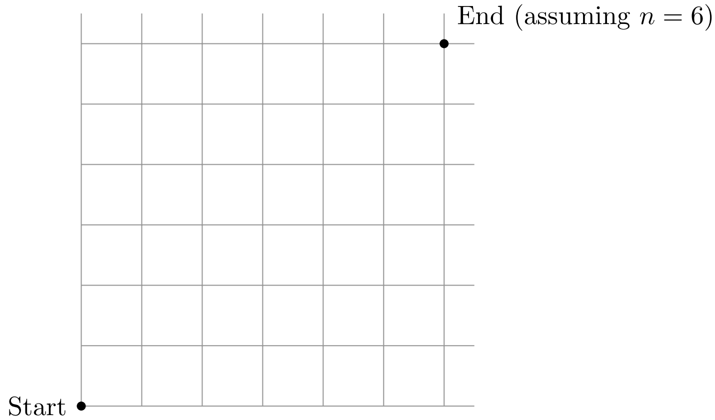
$$a(b+c) = ab + ac$$
$$(a+b)(c+d) = ac + bc + ad + bd.$$

Try to compute the powers of $(1+x)$.

(Hint: Think of $(1+x)^n$ *as* $(1+x)^{n-1}(1+x)$*.)*

n	$(1+x)^n$
0	
1	
2	
3	
4	
5	

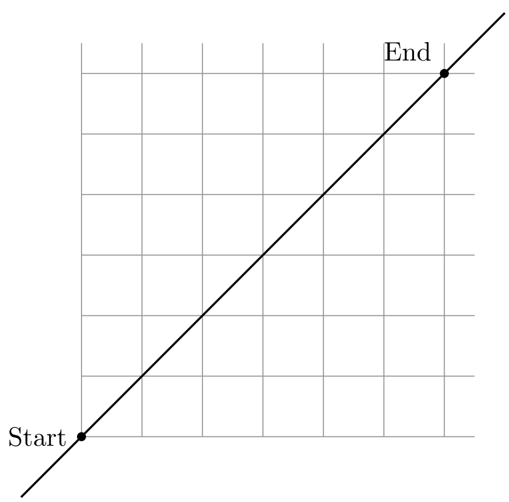
Question 5. You're a taxi driver in a city with a grid-based road layout. You need to drive a passenger from the intersection of 0th St. and 0th Ave. to the intersection of nth St. and nth Ave. How many different paths can you take, if you only ever drive east and north?



n	1	2	3	4	5	6
# paths						

Find a formula for these numbers.

Question 6. It's always good to be prepared for emergencies. What if there was an earthquake that caused a diagonal crack through the city that you couldn't drive across? How many paths are there that stay **above** this line? Again, you're only allowed to travel east and north. You're allowed to touch the line, but you can't cross it.

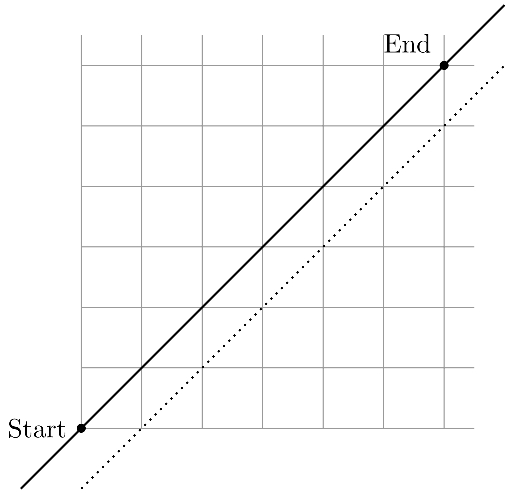


n	1	2	3	4	5	6
# paths						

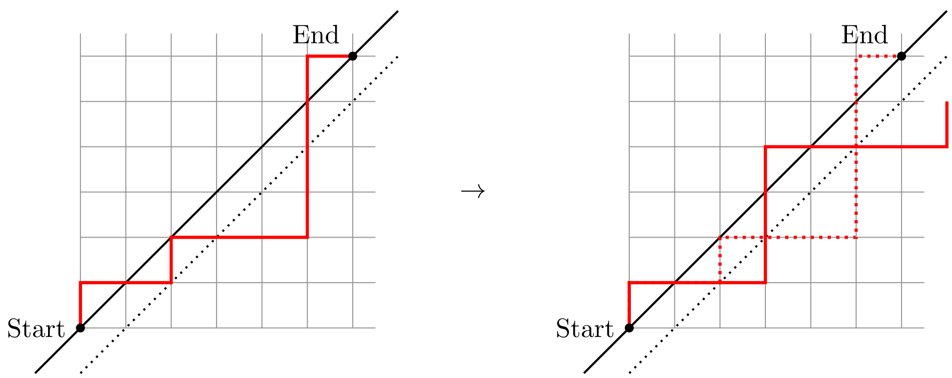
Can you find a formula for these numbers? (If not, turn to the next page.)

Let's try to count these "good" paths that stay above the line, by instead counting the number of "bad" paths that visit the region below the line.

If you draw a "bad" path, it must touch this dotted line:



Draw a "bad" path, and find the **first** time that your path touches the dotted line. Take the **rest** of the path and reflect it across the dotted line, to get a new path. Here's an example:



(a) Draw your own "bad" path and apply this reflecting operation to it.

(b) What's the new end point for this path?

(c) How many paths are there in total from the "Start" point to this new end point? How many of these paths are "bad", and how many are "good"? (Remember, a path is "bad" if it ever goes below the diagonal crack.)

(d) This reflecting operation takes a path from $(0, 0)$ to (n, n) and produces a path from $(0, 0)$ to the new end point. Is there a reverse operation?

What does this imply about the number of "bad" paths from $(0, 0)$ to (n, n)?

(e) What's the total number of paths from $(0, 0)$ to (n, n)? Therefore, how many "good" paths are there?

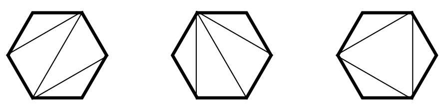
Question 7. Count the following things:

(a) The number of ways to arrange n pairs of opening and closing parentheses, "(" and ")", so that the opening and closing pairs "match up" in the usual way.

(())()   ((()))   ()()()

n	1	2	3	4	5	6
# ways						

(b) The number of ways to subdivide a regular n-gon into triangles by drawing non-overlapping lines between the corners.



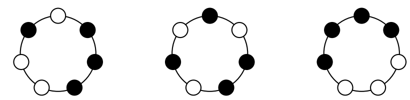
n	3	4	5	6	7	8
# ways						

(c) The number of ways to arrange the first n letters of the alphabet in a list so that no three letters (not necessarily adjacent) are in alphabetical order.

E,D,A,C,B   C,B,E,D,A   D,E,C,A,B

n	1	2	3	4	5	6
# ways						

(d) The number of ways to thread n white beads and $n + 1$ black beads on a necklace. Rotations of a necklace count as the same necklace, but reflections don't.



n	1	2	3	4	5	6
# ways						

(e) The number of ways to divide n people into subgroups.

(AC)(BDE)   (A)(BE)(CD)
(ACDE)(B)

n	1	2	3	4	5	6
# ways						

(f) The number of ways to fill a $2 \times n$ rectangle with the numbers $1, 2, \ldots, 2n$ so that every number is less than the number to its right and the number below it.

1	3	4
2	5	6

1	2	5
3	4	6

1	2	3
4	5	6

n	1	2	3	4	5	6
# ways						

Can you explain any of the connections between these numbers?