# Week 4

Over the last couple of weeks, we've learnt about a couple of meanings for the notation "$\binom{n}{k}$" (pronounced "$n$ choose $k$"). It means:

- The $k$th number in the $n$th row of Pascal's triangle,
- The number of ways to choose a subset of $k$ things out of a set of $n$ things.

Today, we'll investigate a third meaning:

- The formula $\dfrac{n!}{k!(n-k)!}$

(Remember, "$n!$" or "$n$ factorial" means "$1 \times 2 \times 3 \times \cdots \times n$", and it's equal to the number of ways of ordering $n$ things in a list.)

**Question 1.** If you already know this formula, explain to the rest of your group why this formula equals the number of ways to choose $k$ things out of a set of $n$, WITHOUT reading the hint below. Make sure everyone understands before you move on!

If no one in your group has seen this formula, read this hint to try to figure it out yourselves:

*(Hint: Think about choosing the people in a sub-team one at a time. When you choose the first person, you have $n$ choices; when you choose the next person, you have $n-1$ people left to choose from; and so on, until you have $n-k+1$ choices for the $k$th person. Explain why the total number of choices so far is $n!/(n-k)!$.*

*This method has some redundancy, though: for example, if you choose Bruce, Conrad and Egbert, you get the same subteam as choosing Conrad, Egbert and Bruce. How many ways are there to get the same subteam of size $k$?)*

**Question 2.** Now that we know a formula for $\binom{n}{k}$, try to use this formula to explain some of the patterns we've spotted in Pascal's triangle.

(a) $\displaystyle\binom{n}{k} = \binom{n}{n-k}$. (The symmetry in Pascal's triangle.)

(b) $\displaystyle\binom{n}{1} = n$. (The second-from-left diagonal in Pascal's triangle.)

(c) $\displaystyle\binom{n}{0}$ and $\displaystyle\binom{n}{n}$ both equal 1. (The lines of 1s along the edges of Pascal's triangle.)

(d) $\displaystyle\frac{n}{k} \times \binom{n-1}{k-1} = \binom{n}{k}$. (This is a fact we haven't seen before!)

(e) $\displaystyle\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. (The "each number is the sum of the two numbers above it" rule for Pascal's triangle.)

**Question 3.** Let's take a closer look at $\binom{n}{2}$.

(a) Write down a formula for $\binom{n}{2}$ that doesn't use any factorials.

(b) So far, we've only thought about when $n$ is a non-negative integer. Does your formula for $\binom{n}{2}$ work for other kinds of values for $n$?

For example, according to your formula, what are these numbers:

i. $\binom{4.5}{2}$

ii. $\binom{-1}{2}$

iii. $\binom{\pi}{2}$

And if you know about complex numbers:

iv. $\binom{i}{2}$

(c) What do you think of this claim:

$$\binom{-n}{2} = \binom{n+1}{2}$$

(d) Can you do something similar with $\binom{n}{3}$? What do you think about this claim:

$$\binom{-n}{3} = -\binom{n+2}{3}$$

How about $\binom{n}{4}$? Or $\binom{n}{k}$?

**Question 4.** Suppose $p$ is a prime number (that is, a number with no factors except itself and 1). Explain why $\binom{p}{k}$ is always divisible by $p$, when $0 < k < p$.

*(Bonus challenge problem: can you answer this question without using the formula, just the "subteams" interpretation for* $\binom{n}{k}$*?)*

How about $\binom{p^m}{k}$?

**Question 5.** When is $\binom{n}{k}$ *equal* to a prime number $p$?

**Question 6.** Can you explain any other binomial coefficient patterns using this formula, or the "subteams" meaning?

Here are some other patterns (some we've seen before, some new):

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

$$0 \times \binom{n}{0} + 1 \times \binom{n}{1} + 2 \times \binom{n}{2} + \cdots + n \times \binom{n}{n} = n \times 2^{n-1}$$

Can you find any others?