UW Math Circle

Instructions: Saruman forgot what numbers are, so he decided to make up his own. Here are some numbers he invented:

$$123, 32154, 1423, 312, 2413, 162543$$

Gandalf decided to make up some numbers of his own, but Saruman told him that they don't count:

$$22, 243, 35, 541344, 1337$$

1. Which of these numbers count according to Saruman's definition of a number?

   $$615324, 1327645, 31514, 12345, 541, 7, 41532, 3521, 312, 4132, 13542, 1, 14263675$$

2. (a) Write down all the one-digit numbers according to Saruman's rules. How many are there?

   (b) Write down all the two-digit numbers according to Saruman's rules. How many are there?

   (c) Write down all the three-digit numbers according to Saruman's rules. How many are there?

   (d) Write down all the four-digit numbers according to Saruman's rules. How many are there?

   (e) How many n-digit numbers are there according to Saruman's rules?

3. Next, Saruman made up rules for how to multiply two numbers together. Here are some of his results:

$$213 \cdot 321 = 231$$
$$2143 \cdot 4321 = 3412$$
$$21435 \cdot 54321 = 45231$$
$$214356 \cdot 654321 = 563421$$
$$2143567 \cdot 7654321 = 6745321$$
$$21435678 \cdot 87654321 = 78564321$$

$$321 \cdot 213 = 312$$
$$4321 \cdot 2413 = 3142$$
$$54321 \cdot 13524 = 42531$$
$$654321 \cdot 234651 = 156432$$
$$7654321 \cdot 2543671 = 1763452$$

$$1324 \cdot 3421 = 3241$$
$$1324 \cdot 3412 = 3142$$
$$1324 \cdot 3142 = 3412$$
$$1324 \cdot 1342 = 1432$$
$$1324 \cdot 1324 = 1234$$
$$1324 \cdot 1432 = 1342$$

$$123 \cdot 123 = 123$$
$$123 \cdot 321 = 321$$
$$123 \cdot 231 = 231$$
$$132 \cdot 231 = 213$$
$$213 \cdot 213 = 123$$
$$213 \cdot 321 = 231$$
$$213 \cdot 123 = 213$$
$$321 \cdot 321 = 123$$
$$231 \cdot 231 = 312$$
$$231 \cdot 213 = 132$$

$$23145 \cdot 41253 = 12453$$
$$3761542 \cdot 2617534 = 1432576$$

How would Saruman multiply these numbers:

$$312 \cdot 132 \ =$$
$$52143 \cdot 14235 \ =$$
$$21 \cdot 21 \ =$$
$$1347652 \cdot 3614527 \ =$$
$$4321 \cdot 2341 \ =$$

4. Can you find a number so that

$$31425 \cdot \underline{\hspace{2cm}} = 12345$$

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