UW Math Circle
April 9, 2020

Probability number theory problems

Warmups:

1. What is the probability (approximately) that a random integer is equal to 1 mod 5 and also 3 mod 7?
2. What is the probability (approximately) that a random integer is divisible by 2, but *not* by 4? What about the probability of being divisible by 64, but *not* 128?
3. What is the probability (approximately) that a random integer between 1 and 1 million is a square integer? (The square integers are 1, 4, 9, 16, ...)
4. What is the probability (approximately) that a random integer between 1 and 1 million is a cube? (The cubes are 1, 8, 27, ...)
5. What is the probability (approximately) that a random integer between 1 and 1 million is a triangle number? (The triangle numbers are 1, 3, 6, 10, ...)
6. What is the probability (approximately) that a random integer is divisible by exactly one of 2, 3, and 5? Exactly two?

Harder problems:

7. Pick two random integers M and N between 1 and 10^6. What is the probability (approximately) that either M divides N or N divides M?
8. Pick two random integers M and N between 1 and 10^6. What is the probability (approximately) that the gcd of M and N is 2? 3? Any given prime p?
9. Let N be a random integer between 1 and 10^6.
   i. Let p be any prime, and k any nonnegative integer. Show that
   $$\mathbb{P}(p^k | N \text{ and } p^{k+1} \not| N) \approx p^{-k}\left(1 - p^{-1}\right)$$
   ii. Let $X(p)$ be the biggest power of p that divides a random integer N between 1 and 10^6. Use the formula from the previous part to figure out the expected value of $X(p)$.
10. Let N be a random integer between 1 and 10^6. Show that, on average, N has
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$
many distinct prime factors. (This sum, up to primes less than 10^6, is approximately equal to $\log\log 10^6 \approx 2.63$.)