Euler’s formula: Polytopes

Here are some polytopes:

A **polytope** is a 3D shape whose sides (or “faces”) are polygons (squares, triangles, hexagons etc. — they don’t have to be regular). Can you think of any more polytopes?

**Problem 1.** How many polytopes can you find where:

- ...all faces are triangles? Quadrilaterals? Pentagons? Hexagons?

- ...every corner of the shape is in exactly 3 faces? Or 4? Or 5, or 6...?

**Problem 2.** For each polytope, count the numbers of corners, edges and faces. Do you notice any patterns? *(Hint: Try adding the numbers of corners and faces together.)*
Euler’s formula: Planar graphs

Remember that a graph is planar if you can draw it (in the 2D plane) without any of the edges crossing over each other. Here are some planar graphs:

Can you draw some more?

Problem 3. Try to find some planar graphs where:

- all vertices are contained in exactly 3 edges? Or 4, or 5, or 6,…

- every region between the edges has 3 sides? Or 4, 5, etc.…

Problem 4. For each planar graph, count the numbers of vertices, edges and regions\(^1\) Do you notice any patterns?

Problem 5. What’s the connection with polytopes?

\(^1\)Question: Should you count the “outside” of the graph as a region? What do you think?
Euler’s formula: Why is it true?!

On the last worksheet, hopefully you discovered this fact: if you add the numbers of vertices and regions in a (connected) planar graph, then subtract the number of regions, you always get 1 (or 2, depending on whether the “outside” counts as a region). In other words, $V - E + F = 1$, where $V$, $E$ and $F$ stand for the numbers of vertices, edges and regions respectively. In this worksheet, we’re going to figure out why this works!

To begin with, let’s think about trees. A tree is a graph that doesn’t have any cycles — that is, it’s impossible to follow the edges and get back to where you started without retracing your steps.

Problem 6. Explain why the following fact is true: In every tree that has at least 2 vertices, there is some vertex which only touches one edge. (Such a vertex is called a leaf.)

(Hint: Try walking along edges without retracing your steps.)

Problem 7. Now, explain why Euler’s formula works for trees.

(Hint: First, explain why it works for trees with 2 vertices or fewer. Next, if you have a tree with at least 2 vertices, is there something you can do to turn it into a smaller tree?)

Problem 8. Finally, explain Euler’s formula for all planar graphs.

(Hint: If a graph is not a tree, it must contain a cycle. Now, pick an edge in that cycle, and try something similar to what you did in problem 7.)

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2Question: Is it true that every tree is a planar graph?
Euler’s formula: What is it good for?

Now let’s try to use Euler’s formula to prove some stuff!

**Problem 9.** In problem 1 you probably found it difficult to make any polytopes using only hexagons. Let’s figure out why: we’ll imagine that we’ve found a polytope using only hexagons, and we’ll try to write down Euler’s formula for this polytope.

(a) Use the facts that each face has 6 edges and each edge is contained in 2 faces to write an equation relating $E$ and $F$.

(b) Now, each face has 6 corners, and each corner is contained in at least 3 faces. (If you like, you can assume that all corners are in *exactly* 3 faces — it’ll make the calculations simpler.) Write down an equation relating $V$ and $F$.

(c) Finally, substitute all these equations into Euler’s formula. What happens?

**Problem 10.** Making a polytope using just hexagons is rather hard, so let’s throw in some pentagons as well! Let $P$ stand for the number of pentagons, and $H$ the number of hexagons. Try to do some similar calculations to **Problem 9**. What can you say about the number of pentagons you need?