UW Math Circle

Homework

1. If a taxicab can only drive north and east, in how many ways can the cab drive:
   (a) 1 block north and 2 blocks east?
   (b) 1 block north and 3 blocks east?
   (c) 1 block north and 4 blocks east?
   (d) 1 block north and n blocks east?
   (e) 2 blocks north and 2 blocks east?
   (f) 2 blocks north and 3 blocks east?
   (g) 2 blocks north and 4 blocks east?
   (h) 2 blocks north and n blocks east?
   (i) m blocks north and 1 block east?
   (j) m blocks north and 2 blocks east?
   (k) m blocks north and 3 blocks east?
   (l) m blocks north and n blocks east? Can you answer this question without using any algebra?

2. Prove the following identities without using algebraic manipulations.
   (a) $\binom{n}{5} = \binom{n}{n-5}$
   (b) $\binom{n}{5} = \binom{n-1}{5} + \binom{n-1}{4}$
   (c) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
   (d) $3\binom{n}{3} = n\binom{n-1}{2}$
   (e) $k\binom{n}{k} = n\binom{n-1}{k-1}$
   (f) $\binom{8}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2}$
   (g) $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$
   (How does this compare to $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$?)
   (h) What do you think $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k}$ should equal?