1   Complex Numbers and Geometry

1. Give a geometric description of each of the following.

   - Addition of complex numbers;
   - Subtraction of complex numbers;
   - Multiplication;
   - Division;
   - Conjugation;
   - The function $z \to 1/z$.

2. Using Complex numbers, how can you verify if a point is on a line? how can you tell if two lines are parallel? How about orthogonal? How do you find the midpoint of a line segment? What other geometric conditions can you check?

3. Prove that four distinct points z_1, z_2, z_3 and z_4 lie on a circle if and only if

$$\frac{z_1 - z_3}{z_2 - z_3} \div \frac{z_1 - z_4}{z_2 - z_4}$$

   is real. This is called the cross ratio.

4. Use complex geometry to prove that the three medians of a triangle meet at a point - the centroid of the triangle.

5. Given are three disjoint squares $ABCD$, $BEFC$ and $EGHF$. Find the sum of the three angles CAB, FAB and HAB.

6. We build equilateral triangles on the sides of a given triangle (on the outside, so that the three new triangles share only edges with the given triangle). Prove that the centroids of the three new triangles form an equilateral triangle.

7. Let a and b lie on the unit circle and let p be the point of intersection of the tangent lines to the unit circle at a and b. Prove that
$$p = \frac{2ab}{a + b}.$$

8. A quadrilateral $ABCD$ is circumscribed on a circle and the sides AB and CD are parallel. The side DA is tangent to the inscribed circle at E. Point F is the reflection of A with respect to B. The line tangent to the inscribed circle, passing through F and distinct from AB touches the circle at G. Prove that C, E and G are colinear.

2   Complex Numbers and Algebra

1. Find all solutions to the equation $z^n + 1 = 0$. Geometry might help!

2. How does the algebraic structure of the complex numbers compare to the reals? Are they commutative? Associative? Do they have a multiplicative identity? Can you uniquely factor integers into primes? What else?

3. Consider matrices of the form
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
   where a and b are real numbers. How do they relate to complex numbers?

3   Algebra and Geometry!

1. Give a geometric description of the transformation of the complex plane given by

$$z \to az + b$$

   where a and b are complex numbers.

2. Prove that the composition of two rotations of the plane (possibly with different centres) is either a rotation or a parallel translation.

3. Geometrically describe the transformation
$$z \to z^n$$

   for different values of n.

4. Consider the transformation
$$z \to \frac{az + b}{cz + d}$$

   where $ad - bc \neq 0$. Try to describe what such transformations do. Are they defined everywhere?

5. Show that the transformation of the previous problem preserves the cross ratio. What does this tell you about the image of circles? What about lines?

4   More Please!

Consider numbers of the form
$$a + bi + cj + dk$$

where $i^2 = j^2 = k^2 = -1$ and $ijk = -1$.

1. What is ij equal to? What about ji?

2. What properties does this set of numbers have? Associative? Commutative? Does every element have an inverse? Conjugates? A norm?

3. Consider matrices of the form
$$\begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$$

   where a and b are complex numbers. How do they compare to these strange new numbers?