

Problem Set 3 Solutions

UW Math Circle – Advanced Group

Session 5 (24 October 2013)

1. Take small discs D_x and D_y around x and y inside S . We can choose these discs to be of the same radius r . Then any point on any disc of radius r around any point on the segment connecting x and y is on some segment connecting a point in D_x and a point in D_y , so a disc around any point on the segment from x to y is in S and the segment in the interior of S .

2. When we proved Helly's theorem in two dimensions, we first showed that it was true for four sets, then argued by induction. Similarly, here it suffices to show this for three Senators.

Senators 1 and 2 can agree, so either the interval tolerated by 1 is contained in the interval tolerated by 2 (or vice versa) or not.

In the first case, any interval intersecting the interval tolerated by 1 at a point p also intersects the interval tolerated by 2 at point p , so Senator 3 can agree with 1 and 2.

In the second case, suppose Senator 1 can tolerate (a_1, b_1) and Senator 2 can tolerate (a_2, b_2) , where, without loss of generality, $a_1 < b_1 < a_2 < b_2$. The two agree on (b_1, a_2) . If Senator 3 (who can tolerate (a_3, b_3)) cannot agree with the other two, then either $a_3 > a_2$, in which case 3 and 1 cannot agree, or $b_3 < b_1$, in which case 3 and 2 cannot agree. It follows that 3 can agree with 2 and 1.

3. If three of the points lie on a line, then this is trivial. If one of the points is inside the convex hull of the other three, this is also trivial. Otherwise, the four points form a convex quadrilateral. A geometric argument, or one by analytic geometry, shows that its diagonals intersect.

4. **Elegant step-by-step solution** (found by Thomas): Our 100 Senators are standing outside the Galactic Senate Building. They will be led, one by one, into one of three chambers inside the building.

We call a Senator angry if this Senator is inside, but the person at whom he or she threw a space rock is outside. We will lead them into the building in such a way that there is never more than one angry Senator.

At each step we do the following. If there is an angry Senator, choose the person at whom the angry Senator threw a space rock. Else, choose a random person waiting outside. The chosen person (X) can go in any of the three chambers except the one containing the Senator at whom X threw a rock and possibly the one containing the angry person. (Note that because at most one Senator was angry, there is only one Senator inside who threw a rock at X .) So, there is at least one chamber where we can safely put X . As a result, the number of angry people has not risen above 1. We continue doing this until every Senator is inside.