Math Circle - Beyond Pigeonholes

1. A warehouse contains 200 boots of size 41, 200 boots of size 42, and 200 boots of size 43. Of these 600 boots, there are 300 left boots and 300 right boots. Prove that one can find among these boots at least 100 usable pairs.

2. Given any set of ten integers, prove that we can choose some collection of them such that the collection's sum is divisible by 10.

3. Given an ordered list of 101 distinct numbers, prove there exists a way to choose 11 positions in this list whose numbers form either an increasing or decreasing sublist.

4. Prove that if we choose 51 numbers from the set $\{1, 2, \ldots, 100\}$ then we are guaranteed to have two numbers such that one is a multiple of the other.