UW Math Circle, Spring 2013 - Homework 4

Due May 9, 2013

This week we learned about the **geometric series**, a sum that arises very often in mathematics. In its finite form, it looks like this:

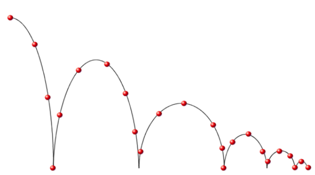
$$a_0 + a_0 r + a_0 r^2 + \ldots + a_0 r^n$$

where a_0 and r are some real numbers (we focused on $a_0 = 1$). In class we showed that this sum evaluates to $a_0 \left(\frac{1-r^{n+1}}{1-r}\right)$. We then looked at the infinite series

$$a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \ldots$$

We showed that this sum becomes infinitely big if $r \geq 1$ or $r < -1$ (think about what happens when $r = -1$!), but if $-1 < r < 1$ then

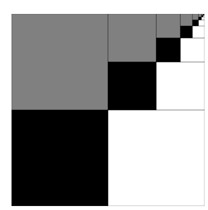
$$a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \ldots = \frac{a_0}{1-r}$$



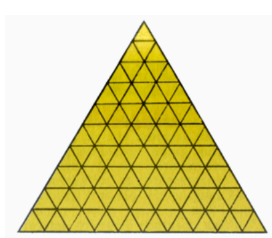
1. Austin has a rubber ball that, when dropped, bounces x% of the way to its original height. He notices that when he drops the ball from 10 feet off the ground, it travels a total vertical distance of 100 feet before stopping. Help Austin determine x.

2. Prove that

$$(1 + x + x^2 + \ldots + x^9)(1 + x^{10} + x^{20} + \ldots + x^{90}) \times (1 + x^{100} + x^{200} + \ldots + x^{900}) \ldots = \frac{1}{1-x}$$



3. Prove that if $x + \frac{1}{x}$ is an integer (whole number) for some x, then $x^n + \frac{1}{x^n}$ is an integer for all n. (*Hint: use induction!*).



4. The sides of an equilateral triangle are marked with n evenly spaced points. Each pair of points are connected by a line parallel to the triangle's sides to form n^2 smaller triangles (see picture). An ant is placed on one of these triangles and starts to move to adjacent triangles without ever visiting the same triangle twice. What is the maximum number of triangles that the ant can visit?