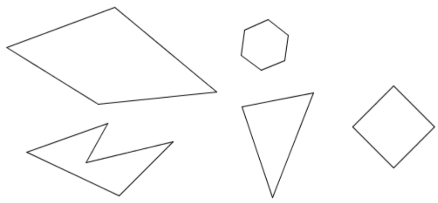
Things to Think on Week 2

1. Everyone knows what polygons are, they're like these:



   If we wanted to be precise we could say something like "A polygon is a chain of non-intersecting line segments where the last segment in the chain meets the first segment." But we don't need to. Anyway: The first problem is to show that, given any polygon, you can always connect two vertices together with a line that lies entirely in the interior of the polygon (thus dividing the original in to two polygons with fewer sides.) Do not assume anything you shouldn't!! (For example, if you know what this means, do not assume the polygons are "convex" or "regular" or special in any way.)

2. Suppose I have a polygon lying on a grid (called a "lattice polygon"), such that all of its vertices lie on points of the grid, like this:

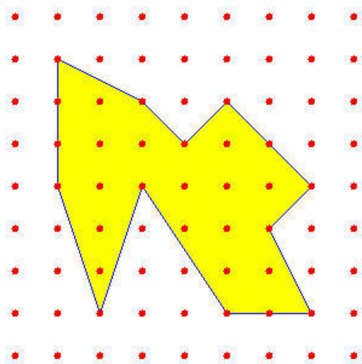


Figure 1: example of a lattice polygon

   Prove that you can compute its are by the following formula:

$$\text{Area} = \{\text{the number of lattice points fully inside}\} + \frac{\{\text{the number of lattice points on the boundary}\}}{2} - 1$$

   Hint: Use problem 1 and induction on the number of vertices, where your base case should be when the number of vertices is 3.

3. Recall that the Fibonacci sequence looks like this:

$$1, 1, 2, 3, 5, 8, 13, ...$$

Where every number is obtained by adding the two preceding it. We can express this using symbols in a very compact way: Let F_k denote the kth Fibonacci number (so $F_1 = 1, F_2 = 1, F_3 = 2$, etc). Then the Fibonacci sequence is the unique sequence of numbers satisfying

$$F_1 = 1, F_2 = 1, \text{and for all } k > 1, F_{k+1} = F_k + F_{k-1}.$$

These formulas should be seen as a way to "build" the sequence recursively- just as how it is usually explained. However, what's nice about phrasing it this way is that we can apply induction more easily: If we want to make a statement about all the Fibonacci numbers, we should prove that the statement is true for F_0 and F_1, and then show that if the statement is true for some F_{k-1} and F_k, it is also true for F_{k+1}; then the result follows by induction! In this problem, we'll prove the following fun fact:

($*$) The only prime Fibonacci numbers are F_4 and F_n when $n > 4$ is prime.

Now at first this might seem very hard, it's not obvious how to inductively prove this directly without treating a bunch of different cases at once. So how should we think about this problem? First let's look at the statement of the claim: why the heck is 4 in there? What's so special about it? As it happens, here's at least one thing that makes 4 special:

(A) Show that any composite (i.e. not prime) number bigger than 4 is divisible by some number bigger than 2.

Ok, so things are vaguely looking up: 4 and the rest of the primes share the property that they aren't divisible by things bigger than 2; also 2 has something to do with Fibonacci numbers, and we can restate ($*$) equivalently as:

($**$) If n is divisible by something bigger than 2, then F_n is composite.

(B) Show that if we know ($**$) is true, then we can prove ($*$) is true.

At this point you should stop reading and go through a few examples yourself: show that F_6, F_8, and F_9 are composite, and see if you notice any special choices of divisors!

Ok, you're back. Maybe you did enough examples to wildly guess that the following statement is true:

($\star$) If $n = km$, then the Fibonacci number $F_n = F_{km}$ is divisible by F_m.

(C) Show that if we know ($\star$), we can prove ($**$), and so can prove ($*$). At this point you should play around a bit more, and when you get back complete the following exercises:

(D) Show (using induction on n) that

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$$

(E) Use (D) and induction on k to prove ($\star$). Recall that $(\star) \Rightarrow (**) \Rightarrow (*)$ so we win! Yay!