BOOK REVIEWS

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Harmonic measure, by John B. Garnett and Donald E. Marshall, Cambridge University Press, 2005, xv + 571 pp., US$110, ISBN 0-521-47018-8

1. Introduction

Harmonic Measure by Garnett and Marshall takes the reader from fundamental concepts to recent research in geometric function theory, an area that has seen many exciting developments over the last 25 years. But before describing the book in detail, I will introduce the topic and describe some of this recent work. We start with a simple formulation of the main problem.

1.1. **Detecting the exit point of Brownian motion.** Consider a random Brownian particle moving in a domain $\Omega \subset \mathbb{R}^2$. We want to find the point where it first hits the boundary, $\partial\Omega$, and to do this we are allowed to place circular detectors along the boundary which register if the particle hits them. If a detector of radius r costs us $\varphi(r)$ (for some increasing φ on $(0,\infty)$), can we detect the exit point almost surely (a.s.) on a finite budget? Note that to detect an exit at x, the point must be contained in infinitely many detectors whose radii tend to zero (Figure 1).

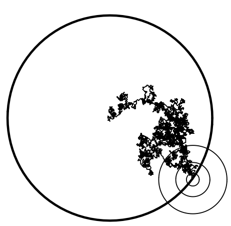

Figure 1.

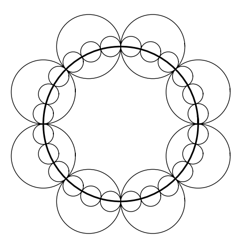

Figure 2.

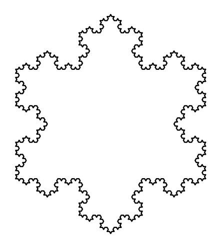

Figure 3.

When Ω is the unit disk, $\mathbb{D}$, and the Brownian particle starts at 0, the hitting distribution on $\mathbb{T} = \partial\mathbb{D}$ is just normalized Lebesgue measure. Thus to detect the exit point a.s., we must cover almost every (a.e.) point of $\mathbb{T}$ by arbitrarily small balls. This implies the sum of the radii of the disks is infinite; so, if $\varphi(r) \geq r$, then we can't detect the exit point on a finite budget. However, if $\varphi(r) = o(r)$ (i.e., $\varphi(r)/r \to 0$ as $r \to 0$), then we can: cover $\mathbb{T}$ by about n_k balls of size $1/n_k$ and let

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$n_k \nearrow \infty$ so fast that $\sum n_k \varphi(1/n_k) < \infty$. See Figure 2. Thus φ "works" for $\mathbb{D}$ iff $\varphi(t) = o(t)$.

If $\partial\Omega$ is the von Koch snowflake (Figure 3) it takes roughly 4^n balls of size 3^{-n} to cover the whole boundary, which we can do on a finite budget iff $\varphi(t) = o(t^\alpha)$, where $\alpha = \log 4/\log 3 > 1$. Thus it seems more expensive to detect exit points on a fractal boundary. However (and this is the main point), not all parts of the snowflake are equally likely to be hit by Brownian motion, and there is a "small" subset of $\partial\Omega$ which still gets hit with probability 1. In fact, it turns out we can detect exit points from the snowflake a.s. with cost function $\varphi(r) = r$, more cheaply than we can do for the circle! To explain why, we will need some notation.

1.2. **The dimension of harmonic measure.** Given $z \in \Omega \subset \mathbb{R}^2$ and $E \subset \partial\Omega$, the *harmonic measure* of E in Ω with respect to z (denoted $\omega(z, E, \Omega)$ or $\omega(E)$ for brevity) is the probability that a Brownian motion started at z first exits Ω in E. For a fixed E it is a harmonic function (hence the name) with values in $[0, 1]$. By the minimum principle, if this function vanishes anywhere on Ω it vanishes everywhere; thus the sets of harmonic measure zero do not depend on z. Can we geometrically characterize these ω-null sets?

This problem is best understood when Ω is simply connected, for then the Riemann mapping theorem provides a 1-1, holomorphic map $f : \mathbb{D} \to \Omega$ and the conformal invariance of Brownian motion implies

$$\omega(f(0), E, \Omega) = \omega(0, f^{-1}(E), \mathbb{D}) = |f^{-1}(E)|/2\pi.$$

Estimates for conformal mappings then imply estimates for harmonic measure with base point $z = f(0)$; e.g., for an interval $I \subset \mathbb{T}$ and $J = f(I) \subset \partial\Omega$, we often have

$$\text{diam}(J) = \text{diam}(f(I)) \approx |f'(z_I)|\text{diam}(I) \approx |f'(z_I)|\omega(J), \tag{1.1}$$

where $z_I = c_I(1 - |I|)$ and c_I is the center of I (think of z_I as the top of a "tent" with base I). The estimate is true as stated for many domains (e.g., quasidisks such as the snowflake) and is true in general if we replace I by a certain subset of smaller measure (which is usually adequate). Here and later, $A \approx B$ means that A and B have bounded ratio.

We want to estimate harmonic measure by comparing it to the more geometrically defined Hausdorff measures: given an increasing function φ on $[0, \infty)$, the Hausdorff φ-measure of E is

$$\mathcal{H}_\varphi(E) = \lim_{\delta \to 0} \inf\{\sum_n \varphi(r_n) : E \subset \cup B(x_n, r_n), r_n \le \delta\}.$$

This measures the "price" of covering E by small balls whose cost is given by φ. When $\varphi(t) = t^\alpha$ we denote this by $\mathcal{H}_\alpha$ ($\alpha = 2$ gives a multiple of Lebesgue area measure; $\alpha = 1$ gives length). The Hausdorff dimension of a set is defined by

$$\dim(E) = \inf\{\alpha : \mathcal{H}_\alpha(E) = 0\}.$$

The smaller α is, the more expensive it is to cover E; the dimension marks the transition from positive to zero cost coverings. The dimension of a measure μ is the smallest dimension of a full μ-measure set, i.e.,

$$\dim(\mu) = \inf\{\dim(E) : \mu(E^c) = 0\} = \inf\{\alpha : \mu \perp \mathcal{H}_\alpha\},$$

where E^c is the complement of E, and we write $\mu \perp \nu$ if there is a set E such that $\mu(E) = \nu(E^c) = 0$. We write $\mu \ll \nu$ if $\nu(E) = 0 \Rightarrow \mu(E) = 0$ and write $\nu \sim \mu$

if $\nu \ll \mu \ll \nu$. We always have $\dim(\mu) \le \dim(\mathrm{supp}(\mu))$, but strict inequality is possible (e.g. point masses on a countable dense set).

How does our problem about the cost of detecting exit points fit into this notation? A collection $\mathcal{C}$ of disks is called a Vitali covering of a set E if for each $\epsilon > 0$, $\mathcal{C}_\epsilon = \{D \in \mathcal{C} : \mathrm{diam}(D) < \epsilon\}$ is also a cover. We can detect a.e. exit point of Brownian motion on a finite φ-budget iff there is a Vitali covering of a full ω-measure set E by disks of radius $\{r_j\}$ such that $\sum_j \varphi(r_j) < \infty$. This happens iff $\mathcal{H}_\varphi(E) = 0$, which happens iff $\omega \perp \mathcal{H}_\varphi$. Conversely, $\omega \ll \mathcal{H}_\varphi$ holds iff any set E which we can afford to test has zero ω-measure. Thus the detection question is really: for what functions φ is $\omega \perp \mathcal{H}_\varphi$? When is $\omega \ll \mathcal{H}_\varphi$?

If we can find a φ so that $\omega \sim \mathcal{H}_\varphi$, then we can claim to "completely understand" ω. One case when this happens is when $\partial\Omega$ has finite length:

Theorem 1.1 ([F. Riesz and M. Riesz (1916)])**.** $\mathcal{H}_1(\partial\Omega) < \infty \Rightarrow \omega \sim \mathcal{H}_1$.

This is a perfect example of what we want: a geometric criterion (zero 1-measure) which exactly identities the ω-null sets. We don't have $\omega \sim \mathcal{H}_1$ for general simply connected domains, but surprisingly ω is always "1-dimensional":

Theorem 1.2 ([Makarov(1985)])**.** *If Ω is simply connected, then* $\dim(\omega) = 1$*, i.e., $\omega \ll \mathcal{H}_\beta$ for every $\beta < 1$ and $\omega \perp \mathcal{H}_\alpha$ for all $\alpha > 1$.*

Makarov proved an even more precise version (one with lots of logarithms) which we will discuss later. This unexpected, elegant and influential theorem earned him the 1986 Salem Prize, and I will try to describe his results and how they fit into what came before and after.

2. Harmonic measure is at most one dimensional

2.1. **Plessner, McMillan and Pommerenke.** The "right" way to take boundary values of a holomorphic function in the disk is non-tangentially: i.e., as $z \to w \in \mathbb{T}$ through a cone in $\mathbb{D}$ centered along a radius, Figure 4. The most drastic way the limit could fail to exist is if f is non-tangentially dense (i.e., $f(z)$ can approach any complex value along some non-tangential sequence approaching w). In turns out that, except on zero measure, one of the two extremes must happen:

Theorem 2.1 (Plessner's theorem)**.** *If f is analytic on $\mathbb{D}$, then at Lebesgue a.e. point of $\mathbb{T}$, either f has a non-tangential limit or is non-tangentially dense.*

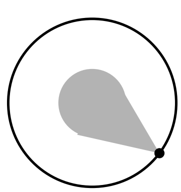

Figure 4.

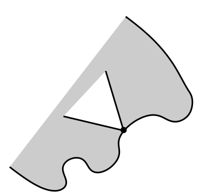

Figure 5.

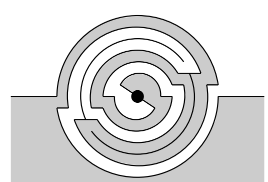

Figure 6.

A boundary point of a simply connected domain Ω is called a *cone point* if it is the vertex of a cone in Ω (Figure 5) and is called an *inner tangent* if the angle of this cone can be taken as close to π as we wish, but no larger. These sets will be denoted Cone(Ω), IT(Ω). If Γ is a closed Jordan curve with complementary components Ω_1, Ω_2, its set of tangent points is $\mathrm{Tan}(\Gamma) = \mathrm{IT}(\Omega_1) \cap \mathrm{IT}(\Omega_2)$. A boundary point x is called a *twist point* if

$$\limsup \arg(z - x) = +\infty, \quad \liminf \arg(z - x) = -\infty,$$

as z approaches x in Ω. The set of twist points is denoted $\mathrm{Tw}(\Omega)$; they are hard to visualize, but are basically points where $\partial\Omega$ spirals around x in both directions (imagine Figure 6 continued for infinitely many scales with an increasing number of twists). This sounds like it might be rare, but according to McMillan's twist point theorem, a.e. boundary point is either an inner tangent or a twist point:

Theorem 2.2 ([McMillan(1969)]). $\omega(\mathrm{IT}(\Omega) \cup \mathrm{Tw}(\Omega)) = 1$.

Moreover, if $f : \mathbb{D} \to \Omega$ is conformal, McMillan showed these two sets correspond to the sets in Plessner's theorem where f' has non-tangential limits or is non-tangentially dense and that $\omega \sim \mathcal{H}_1$ on the inner tangents. Thus any failure of the Riesz theorem for general domains must occur on the twist points. In fact, it *always* fails on the twist points, for [Pommerenke(1986)] refined an argument of Makarov to show that ω-a.e. on $\mathrm{Tw}(\Omega)$,

$$\limsup_{r \to 0} \frac{\omega(B(x, r))}{r} = \infty. \tag{2.1}$$

(One often writes $\omega(\partial\Omega \cap B)$ instead of $\omega(B)$ for $B = B(x, r)$, but since ω is supported on $\partial\Omega$, the two are equal and the latter is more compact.) Equation (2.1) says ω is more "compressed" than $\mathcal{H}_1$ and it implies

Theorem 2.3. *$\omega \perp \mathcal{H}_1$ on* $\mathrm{Tw}(\Omega)$.

(2.1) holds because at a.e. $x \in E = f^{-1}(\mathrm{Tw}(\Omega))$, f' is non-tangentially dense; thus there is a non-tangential sequence converging to x along which $f' \to 0$ and hence $|f'| < \epsilon$ eventually. By (1.1) these points correspond to arcs on $J \subset \partial\Omega$ where $\omega(J) > \mathrm{diam}(J)/\epsilon$. Taking $\epsilon \to 0$ gives (2.1). Next, (2.1) and Vitali's theorem (every Vitali cover contains a pairwise disjoint subcover of a.e. point) imply there is a set $E \subset \mathrm{Tw}(\Omega)$ so that $\omega(E) = \omega(\mathrm{Tw}(\Omega))$ and for any $\epsilon > 0$, E can be covered by disjoint balls $B_j = B(x_j, r_j)$ with $r_j \le \epsilon\omega(B_j)$. Since $\sum \omega(B_j) \le 1$, $\mathcal{H}_1(E) = 0$ and so Theorem 2.3 holds.

2.2. **Some examples.** We can now see why both directions of the F. and M. Riesz theorem fail in general. First, Figure 7 shows a domain whose bottom edge E contains no inner tangents or twist points; thus $\mathcal{H}_1(E) > 0$ but $\omega(E) = 0$. Second, for the snowflake domain one can prove directly that inner tangents have zero $\mathcal{H}_1$ measure (they all lie on a union of Cantor sets of dimension < 1), so by McMillan's theorem $\omega(\mathrm{Tw}(\Omega)) = 1$. By Pommerenke's theorem $\omega \perp \mathcal{H}_1$, i.e., there is a set with $\mathcal{H}_1(E) = 0$ but $\omega(E) > 0$. The first such example was given by [Lavrent′ev(1936)] and a simpler example was given by [McMillan and Piranian(1973)].

Consider the two harmonic measures ω_1, ω_2, corresponding to opposite sides of a Jordan curve Γ. By McMillan's theorem, on $\mathrm{Tan}(\Gamma)$ we have $\omega_1 \sim \omega_2 \sim \mathcal{H}_1$. A famous estimate of Ahlfors implies $\omega_1(B)\omega_2(B) = O(r^2)$ for any ball $B = B(x, r)$. Using this and (2.1), we can prove $\omega_1 \perp \omega_2$ on $\mathrm{Tw}(\Gamma)$ (at scales when $\omega_1(B) \gg r$ (2.1) implies $\omega_2(B) \ll r$ so the two measures can't be comparable). Indeed,

$$\omega_1 \perp \omega_2 \Leftrightarrow \mathcal{H}_1(\mathrm{Tan}(\Gamma)) = 0.$$

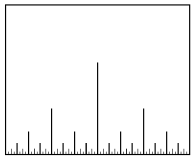

FIGURE 7.

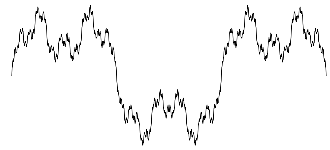

FIGURE 8.

See [Bishop et al.(1989)]. Thus for a snowflake, Brownian particles on opposite sides are absorbed a.s. by disjoint subsets of Γ.

The Weierstrass function $f_{a,b}(z) = \sum_{n=1}^{\infty} a^{-n} \sin(b^n x)$ with $a = 2, b = 3$ is shown in Figure 8. If $b > a \geq 2$, then f has no finite or infinite derivatives, so its graph has no tangents (not even vertical ones), and hence the inner tangents for the two sides are disjoint. Hence $\omega_1 \perp \omega_2$. However, since a graph has no twist points, $\omega_i(\mathrm{IT}(\Omega_i)) = 1$, $i = 1, 2$. When $a = b$, the Weierstrass function has no finite derivatives, but it is in the Zygmund class (so the graph is a quasicircle), which implies $\mathrm{Tan}(\Gamma) = \mathrm{IT}(\Omega_i)$, $i = 1, 2$ and $\mathrm{Tan}(\Gamma)$ has full ω-measure from both sides. Since f has no finite derivatives, every tangent line must be vertical. This implies $\omega_1 \sim \omega_2 \sim \mathcal{H}_1$ and "live" on a set which projects vertically to a set of zero length.

It is possible to construct a curve Γ so that $\omega_1(\mathrm{Tw}(\Gamma)) = 1$ and $\omega_2(\mathrm{Tw}(\Gamma)) = 0$ (but I leave this as an exercise).

2.3. **The local F. and M. Riesz theorem.** Given (2.1) and the dichotomies in Plessner's and McMillan's theorems, it is natural to ask if (e.g., [Bishop(1992)])

$$\liminf_{r \to 0} \frac{1}{r} \omega(B(x, r)) = 0, \tag{2.2}$$

at ω-a.e. twist point. This was recently proven by [Choi(2004)]. In addition to its intrinsic elegance, Choi's theorem gives an alternate proof of

Theorem 2.4 ([Bishop and Jones(1990)])**.** *If Ω is simply connected and Γ is a rectifiable curve, then $\omega \ll \mathcal{H}_1$ on $\Gamma \cap \partial\Omega$.*

We call this the "local F. and M. Riesz theorem" since it gives half of Theorem 1.1 when $\Gamma = \partial\Omega$. In terms of the "detector problem" at the beginning of this review, this theorem says our detectors must be battery powered; if they have to be connected by a wire with cost proportional to length, then we can never afford to detect exits on twist points, regardless of the cost of the detectors themselves.

The local Riesz theorem follows from Choi's theorem because (2.2) says that ω-a.e. point of $E = \Gamma \cap \mathrm{Tw}(\Omega)$ can be Vitali covered by balls such that $\omega(B(x, r)) \leq \epsilon r$. Taking a disjoint sub-covering, $\{B_n\}$, we get

$$\omega(E) \leq \omega(\cup B_n) \leq \epsilon \sum_n r_n \leq \epsilon \mathcal{H}_1(\Gamma) \to 0.$$

The original proof of the local Riesz theorem is a rather involved combination of Fuchsian groups, covering maps, quadratic characterizations of rectifiability and estimates for the Schwarzian derivative, but it does give a useful quantitative version: $E \subset \partial\Omega \cap \Gamma, \mathcal{H}_1(E) \leq \delta \Rightarrow \omega(E) \leq \epsilon$ where $\delta = \delta(\epsilon, \mathcal{H}_1(\Gamma))$.

3. Harmonic measure is at least one dimensional

3.1. **Beurling, Carleson and Makarov.** We have seen that harmonic measure is at most 1-dimensional for simply connected domains, but why can't it live on a set of smaller dimension? If ω gives positive mass to an α-dimensional set, then we must have $\omega(B) \geq C\text{rad(B)}^\alpha$ for many small balls. However, this is impossible if α is too small. For example, Beurling proved that for any simply connected domain and any ball $B = B(x,r)$ we have $\omega(B) \leq O(\sqrt{r})$. This implies $\omega \ll \mathcal{H}_{1/2}$, so $\dim(\omega) \geq \frac{1}{2}$ for all simply connected domains.

Beurling's estimate is sharp because $\omega(B) \approx \sqrt{r}$ if B is centered at the tip of a spike pointed into the domain. Such points have harmonic measure zero, and we expect a better estimate to hold on sets of positive ω-measure. Indeed, [Carleson(1973)] proved there is an $\epsilon > 0$ so that $\omega(B) \leq O(r^{\frac{1}{2}+\epsilon})$ at ω-a.e. point and hence $\omega \ll \mathcal{H}_{\epsilon+1/2}$. This used a difficult argument for a relatively modest improvement, and there was no further progress until Makarov's striking jump to "$\dim(\omega) = 1$". Indeed, [Makarov(1985)] gave an incredibly sharp description of how small the support of harmonic measure can be:

Theorem 3.1 (Makarov's Law of the Iterated Logarithm (LIL) for ω)**.** *Let*

$$\varphi_C(t) = t \exp\left(C \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}} \right).$$

Then there is a value A so that $\omega \ll \mathcal{H}_{\varphi_A}$ for every simply connected domain and a value B so that $\omega \perp \mathcal{H}_{\varphi_B}$ for some simply connected domain.

Since $\varphi_C(t)$ tends to zero more slowly than t but faster than $t^{1-\epsilon}$ for any $\epsilon > 0$, we can infer that $\omega \ll \mathcal{H}_\beta$ for any $\beta < 1$. Makarov's original examples of sharpness were based on lacunary power series, but [Jones(1989)] gave a geometric criterion which shows that the snowflake domain is such an example, and [Przytycki et al.(1989)] showed "most" rational Julia sets are also examples.

The *Bloch space* is the Banach space of analytic functions on $\mathbb{D}$ with

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|) < \infty.$$

Standard results imply that $\|\log f'\|_{\mathcal{B}} \leq 2 \Rightarrow$ f is conformal $\Rightarrow \|\log f'\|_{\mathcal{B}} \leq 6$, and using (1.1), Makarov's LIL reduces to showing that for $g \in \mathcal{B}$,

$$\limsup_{r\to 1} \frac{|g(re^{i\theta})|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} = O(\|g\|_{\mathcal{B}}) \text{ for a.e. } \theta. \tag{3.1}$$

All the logarithms may seem a bit intimidating, but I will try to convince you they are perfectly natural, indeed, "obvious" in hindsight. Consider the standard random walk S_n on the integers. After n steps, we expect that $|S_n| \approx \sqrt{n}$, but occasionally it is larger, as described by the law of the iterated logarithm (Kolmogorov, Khinchin)

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log n}} = 1, \qquad \text{a.s.}$$

If we set $n = \log\frac{1}{1-r}$ we get the same expression as in Makarov's theorem, and this is not a coincidence nor even a heuristic analog; both follow from a more general LIL for dyadic martingales.

3.2. **Dyadic martingales.** Let $\mathcal{D}_n$ denote the dyadic intervals of generation n in $[0, 2\pi) = \mathbb{T}$, i.e., intervals of the form $\pi[j2^{-n}, (j+1)2^{-n})$. We say $\{S_n\}$ is a dyadic martingale if $S_n = \sum_{k=1}^n X_k$ where X_k is constant on intervals in $\mathcal{D}_k$ and has mean value zero over every interval in $\mathcal{D}_{k-1}$. We let $\langle S\rangle_n = \sum_{k=1}^n X_k^2$ denote its quadratic variation. The most basic example is to let $X_n(x) = \text{sign}(\sin(2^n\pi x))$ be the nth Radamacher function. Then $S_n(x)$ represents the standard random walk on the integers (where step direction at time k depends on the value of the kth binary digit of x). A martingale is called a Bloch martingale if the jumps of S_n are uniformly bounded.

Theorem 3.2 (LIL for dyadic martingales)**.** *If $\{S_n\}$ is a Bloch martingale, then*

$$\limsup_n \frac{|S_n|}{\sqrt{2\langle S\rangle_n \log\log\langle S\rangle_n}} = 1,$$

a.e. on $\{x : \langle S\rangle_\infty = \infty\}$.

Makarov originally proved his LIL by an analytical argument that mimicked the martingale case, but later showed it can be deduced from that case using:

Theorem 3.3. *If $u \in \mathcal{B}$ and $I \in \mathcal{D}_n$, then $u(I) = \lim_{r\to 1} \frac{1}{|I|}\int_I u(rw)dw$ exists and defines a dyadic Bloch martingale on $\mathbb{T}$ so that $|u(I) - u(z_I)| = O(\|u\|_{\mathcal{B}})$.*

Thus the a.e. radial behavior of Bloch functions is the same as the a.e. growth of Bloch martingales. Note that if $I \in \mathcal{D}_n$ then $\log(1-|z_I|)^{-1} \approx n$. Also, if $\{S_n\}$ is Bloch, then $\langle S\rangle_n = O(n)$, so

$$\limsup_n \frac{|S_n|}{\sqrt{n\log\log n}} = O(1),$$

and together these facts imply (3.1).

3.3. **Brennan and Hayman-Wu.** Clearly a conformal map $f : \mathbb{D} \to \Omega$ satisfies $|f'| > 0$ (since otherwise it would not be 1-to-1 near a critical point). If it satisfied the stronger uniform condition $|f'| > \epsilon > 0$ on $\mathbb{D}$, then $\omega \ll \mathcal{H}_1$ would be trivial by (1.1). Makarov's theorem is basically an estimate of how close and how often $|f'|$ can be near 0. Many of the most interesting questions of geometric function theory deal with such estimates in one way or another or with the equivalent formulation: if $g = f^{-1} : \Omega \to \mathbb{D}$, then how close is $|g'|$ to being bounded?

One of the most famous such questions is Brennan's conjecture, which claims that $|g'| \in L^p(dxdy, \Omega)$ for every $p < 4$. This is a special case of the integral means conjecture which concerns the function $B(t)$ defined by

$$B(t) = \sup\{\beta_\varphi(t) : \varphi \text{ is univalent }\}, \text{ where}$$

$$\beta_\varphi(t) = \inf\{\beta : \int |\varphi'(re^{i\theta})|^t d\theta = O(\frac{1}{1-r})^\beta\}.$$

One can show Brennan's conjecture is equivalent to $B(-2) = 1$, and many other famous problems are embedded in evaluating $B(t)$ at various values (e.g., Carleson and Jones showed $B(1) = 1/4$ is equivalent to a natural growth bound for level lines of Green's function and to a Bieberbach type conjecture on the decay of Taylor coefficients). The Brennan-Carleson-Jones-Kraetzer conjecture states that $B(t) = t^2/4$ for $-2 \le t \le 2$. This is the object of intense theoretical and numerical investigation but is still open.

Another sense in which $|g'|$ is "almost bounded" is the Hayman-Wu theorem, which states that $\sup_{\Omega,g,L} \int_{L\cap\Omega} |g'|ds < \infty$; i.e., if L is any line, then the length of $g(L\cap\Omega) \subset \mathbb{D}$ is bounded independent of everything. The Hayman-Wu theorem has been generalized in various directions; e.g., $\int_{\Gamma\cap\Omega} |g'|^p ds < \infty$ if Γ is Ahlfors regular (i.e., $\sup_{x,r} \frac{1}{r}\mathcal{H}_1(\Gamma \cap B(x,r)) \leq C < \infty$) and $p > 1$ depends on C. This is due to to [Walden(1994)], but builds on many previous results: [Kaufman and Wu(1982)], [Fernández et al.(1989)], and [Bishop and Jones(1990)] among others. In particular, it uses the quantified version of the local F. and M. Riesz theorem mentioned earlier and so neatly fits into the rest of our story.

A third example of how $|g'|$ is "almost bounded" comes from [Bishop(2002)]: there is a universal $K < \infty$ so that every simply connected domain Ω can be mapped to $\mathbb{D}$ by a K-quasiconformal map with bounded derivative. Thus the conformal map is always uniformly close to a bounded derivative map in the quasiconformal sense.

4. Beyond the simply connected case

4.1. **Multiply connected plane domains.** For multiply connected domains, one cannot expect to have $\omega \ll \mathcal{H}_\alpha$ for all $\alpha < 1$, since the entire boundary of Ω could have dimension < 1, e.g., if Ω is the complement of a "small" Cantor set. However, it is still true that $\dim(\omega) \leq 1$ for any planar domain. In the case of certain self-similar Cantor sets, this was first proved by [Carleson(1985)], and his ideas were generalized to all domains in [Jones and Wolff(1988)]. Later, "$\dim(\omega) \leq 1$" was improved to "ω gives full mass to a set of sigma finite length" by [Wolff(1993)]. The key idea in the multiply connected case is the equality

$$\int_{\partial\Omega} \log \frac{\partial G}{\partial n} d\omega = C + \sum_{z:\nabla G(z)=0} G(z), \tag{4.1}$$

where C is the Robin constant of $\partial\Omega$ and G is the Green function. Since $G > 0$ on Ω, the right-hand side is always bounded below; hence so is the integral on the left. Thus most of the harmonic measure lives on the set where the logarithm term is bounded below. Since $d\omega = \frac{\partial G}{\partial n} ds$, this means that ω has density bounded below compared to arclength, and this implies that ω lives on a 1-dimensional set (although there are formidable technical details to overcome to make this idea into a proof). If $\partial\Omega$ has "many, well-separated" components in a precise sense (e.g., it is a self-similar Cantor set), then G has "lots" of critical points, and Jones and Wolff proved that the series on the right side of (4.1) diverges at a certain rate. This forces ω to be even more compressed; indeed, $\dim(\omega) < 1$ for such domains.

4.2. **Higher dimensions.** In higher dimensions, not much is known about analogs of Makarov's theorems. The most obvious conjecture would be that for any domain in $\mathbb{R}^n$, $\dim(\omega) \leq n - 1$; [Bourgain(1987)] showed $\dim(\omega) \leq n - \epsilon_n < n$. However, a celebrated example of [Wolff(1995)] shows that $\dim(\omega) > n-1$ is possible for certain "snowball" domains. One of the key ingredients of the proof of Makarov's theorems is that if u is harmonic in the plane, then $\log|\nabla u|$ is subharmonic. This is false in higher dimensions (and its failure is the starting point of Wolff's counterexample), but there is a substitute: if u is harmonic, then $|\nabla u|^{(n-2)/(n-1)}$ is subharmonic. Perhaps $\dim(\omega) \leq n - 1 + \frac{n-2}{n-1}$ for domains in $\mathbb{R}^n$? At present, nothing is known. Wolff snowballs have been further studied by [Lewis et al.(2005)].

5. The book

Over the last 20 years I have often been asked to suggest a "good place to learn about harmonic measure," and from now on the book of Garnett and Marshall will be my first suggestion. It's a great place for graduate students to learn an important area from the foundations up to the research frontier or for experts to locate a needed result or reference. Almost all the topics we have touched on, and many more, are discussed with greater depth and clarity than I have been able to achieve here, and since you are obviously quite interested in harmonic measure (after all, you have read this far), I recommend you get the book for the full story.

The first four chapters deal with "preliminary" material such as conformal maps, the hyperbolic metric, the Hayman-Wu theorem, Green's functions and the Dirichlet problem on general domains and extremal distance, including the famous estimate of Ahlfors involving integrals of the form $\int dt/\theta(t)$. Chapter V contains many interesting applications of material in the first four chapters, including Alhfors' solution of the Denjoy problem, the Teichmüller Modulsatz, estimates of harmonic measure and the first discussion of the angular derivative problem.

The next three chapters deal with the geometric properties of harmonic measure on simply connected domains and covers many of the results mentioned above. Chapter VI has the F. and M. Riesz theorem, as well as the theorems of Plessner, McMillan, Makarov and Pommerenke. Chapter VII deals with the special case of quasidisks (such as the von Koch snowflake) and has numerous results dealing with the Bloch space, A_p weights, chord-arc curves and BMO domains. Chapter VIII covers Makarov's LIL and recent developments on Brennan's conjecture.

Chapter IX deals with infinitely connected domains (including previously unpublished results of Jones and Wolff), and Chapter X covers work of Peter Jones and of this reviewer using rectifiability, square functions and Schwarzian derivatives to prove the most general versions of the Hayman-Wu and F. and M. Riesz theorems. Each chapter ends with bibliographical notes and many exercises. Some of these are actual exercises; others present additional theorems, examples and conjectures.

The choice of what material to include is obviously a matter of personal preference, but the authors could hardly have chosen better. While many interesting topics have to be omitted because of space limitations (e.g., harmonic measure on Julia sets or the higher dimensional results of Bourgain and Wolff), the authors touch on all of the most interesting results for general planar domains. I should point out that the first author is my academic grandfather and the second an academic uncle; thus our mathematical tastes may not be entirely uncorrelated.

The book is well organized and well written. Indeed, the first author won the 2003 Steele Prize for mathematical exposition for his earlier book, *Bounded Analytic Functions*, and this one reaches the same high standard of style and relevance. It deserves a large audience because this material is fundamental to modern complex analysis and has important connections to probability, dynamics, functional analysis and other areas. It will be of immense value to both expert practitioners and students. This is one of a handful of books I keep on my desk (rather than up on a shelf), and I often look through its pages to educate or entertain myself. It is an illuminating survey of the geometric theory of harmonic measure as it stands today and is sure to become a respected textbook and standard reference that will profoundly influence the future development of the field.

References

[Bishop(1992)] C. J. Bishop. Some questions concerning harmonic measure. In *Partial Differential Equations with Minimal Smoothness and Applications (Chicago, IL, 1990)*, volume 42 of *IMA Vol. Math. Appl.*, pages 89–97. Springer, New York, 1992. MR1155854 (93f:30023)

[Bishop(2002)] C. J. Bishop. Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. *Ark. Mat.* 40(1):1–26, 2002. MR1948883 (2003i:30063)

[Bishop et al.(1989)] C. J. Bishop, L. Carleson, J. B. Garnett, and P. W. Jones. Harmonic measures supported on curves. *Pacific J. Math.* 138(2):233–236, 1989. MR0996199 (90d:30069)

[Bishop and Jones(1990)] C. J. Bishop and P. W. Jones. Harmonic measure and arclength. *Ann. of Math. (2)* 132(3):511–547, 1990. MR1078268 (92c:30026)

[Bourgain(1987)] J. Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. *Invent. Math.* 87(3):477–483, 1987. MR0874032 (88b:31004)

[Carleson(1973)] L. Carleson. On the distortion of sets on a Jordan curve under conformal mapping. *Duke Math. J.* 40:547–559, 1973. MR0330430 (48:8767)

[Carleson(1985)] L. Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 10:113–123, 1985. MR0802473 (87b:31002)

[Choi(2004)] S. Choi. The lower density conjecture for harmonic measure. *J. Anal. Math.* 93:237–269, 2004. MR2110330 (2005k:30049)

[Fernández et al.(1989)] J. L. Fernández, J. Heinonen, and O. Martio. Quasilines and conformal mappings. *J. Analyse Math.* 52:117–132, 1989. MR0981499 (90a:30017)

[Jones(1989)] P. W. Jones. Square functions, Cauchy integrals, analytic capacity, and harmonic measure. In *Harmonic Analysis and Partial Differential Equations (El Escorial, 1987)*, volume 1384 of *Lecture Notes in Math.*, pages 24–68. Springer, Berlin, 1989. MR1013815 (91b:42032)

[Jones and Wolff(1988)] P. W. Jones and T. H. Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Math.* 161(1-2):131–144, 1988. MR0962097 (90j:31001)

[Kaufman and Wu(1982)] R. Kaufman and J. M. Wu. Distortion of the boundary under conformal mapping. *Michigan Math. J.* 29(3):267–280, 1982. MR0674280 (84b:31003)

[Lavrent′ev(1936)] M. Lavrent′ev. Boundary problems in the theory of univalent functions. *Math. Sb. (N.S.)* 43(1):815–846, 1936. MR0155970 (27:5903)

[Lewis et al.(2005)] J. L. Lewis, G. C. Verchota, and A. L. Vogel. Wolff snowflakes. *Pacific J. Math.* 218(1):139–166, 2005. MR2224593

[Makarov(1985)] N. G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc. (3)* 51(2):369–384, 1985. MR0794117 (87d:30012)

[McMillan(1969)] J. E. McMillan. Boundary behavior of a conformal mapping. *Acta Math.* 123:43–67, 1969. MR0257330 (41:1981)

[McMillan and Piranian(1973)] J. E. McMillan and G. Piranian. Compression and expansion of boundary sets. *Duke Math. J.* 40:599–605, 1973. MR0318492 (47:7039)

[Pommerenke(1986)] C. Pommerenke. On conformal mapping and linear measure. *J. Analyse Math.* 46:231–238, 1986. MR0861701 (88a:30055)

[Przytycki et al.(1989)] F. Przytycki, M. Urbański, and A. Zdunik. Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps. I. *Ann. of Math. (2)* 130(1):1–40, 1989. MR1005606 (91i:58115)

[Riesz and Riesz(1916)] F. Riesz and M. Riesz. Über die ranwerte einer analytischen Funcktionen. *Quartriéme Congress des Math. Scand.*, pages 27–44, 1916.

[Walden(1994)] B. L. Walden. L^p-integrability of derivatives of Riemann mappings on Ahlfors-David regular curves. *J. Anal. Math.* 63:231–253, 1994. MR1269221 (94m:30022)

[Wolff(1993)] T. H. Wolff. Plane harmonic measures live on sets of σ-finite length. *Ark. Mat.* 31(1):137–172, 1993. MR1230270 (94d:31002)

[Wolff(1995)] T. H. Wolff. Counterexamples with harmonic gradients in $\mathbf{R}^3$. In *Essays on Fourier Analysis in Honor of Elias M. Stein (Princeton, NJ, 1991)*, volume 42 of *Princeton Math. Ser.*, pages 321–384. Princeton Univ. Press, Princeton, NJ, 1995. MR1315554 (95m:31010)

Christopher J. Bishop
SUNY Stony Brook
E-mail address: bishop@math.sunysb.edu