Angular Derivatives and Lipschitz Majorants

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ABSTRACT: It is an old problem, dating back at least to Ahlfors [1930], to give geometric conditions on the boundary of a simply connected domain $\Omega$ near $\zeta \in \partial \Omega$ so that a conformal map $\psi$ of $\Omega$ onto the unit disk or half plane extends to be "conformal" at $\zeta$, in the sense that $\psi$ has a non-zero angular derivative at $\zeta$. In this paper, we solve a problem of Burdzy [1986] and [1987, page 164] by giving geometric conditions for a certain class of regions. The main results, Theorem 9(ii) and Theorem 13(ii) are complementary to Burdzy's work in [1986], and extend earlier work of Rodin and Warschawski [1977]. We also give a classical analysis proof of Burdzy's Theorem in Theorem 13(i). The history of this subject is extensive. The interested reader might begin with Warschawski [1967], Rodin-Warschawski [1977], Baernstein [1988], and the references therein. We will first review material needed to understand our result.

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§1 Background.

Throughout this paper, $\Omega$ will denote a simply connected domain in the complex plane and a map defined on $\Omega$ will be called conformal if it is one-to-one and analytic.

**Definition 1.** We say that $\partial \Omega$ has an inner tangent at $\zeta \in \partial \Omega$ if there is an angle $\theta_0$ so that for every $\beta \in (0, \pi/2)$ there is an $\varepsilon = \varepsilon(\beta) > 0$ and so that the truncated cone

$$\Gamma_{\beta}^\varepsilon(\zeta) = \{ z : |\arg(z - \zeta) - \theta_0| < \beta, \ 0 < |z - \zeta| < \varepsilon \}$$

is contained in $\Omega$. When $\theta_0 = \pi/2$ we say that $\Gamma$ has a vertical inner normal.

**Definition 2.** If $\partial \Omega$ has an inner tangent at $\zeta \in \partial \Omega$ and if $\varphi$ is a conformal map defined on $\Omega$, then we say $\varphi$ is semi-conformal at $\zeta$ if $\varphi$ has a non-tangential limit

$$\varphi(\zeta) \equiv \lim_{\Gamma_{\beta}^\varepsilon(\zeta) \ni z \rightarrow \zeta} \varphi(z)$$

for every $\beta \in (0, \pi/2)$ and if

$$A(\zeta) = \lim_{\Gamma_{\beta}^\varepsilon(\zeta) \ni z \rightarrow \zeta} \arg \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}$$

exists for every $\beta \in (0, \pi/2)$.

If $\varphi$ is semi-conformal at $\zeta$ then for $|\alpha - \theta_0| < \pi/2$, the image of the ray $\{ z : \arg(z - \zeta) = \alpha \}$ is asymptotic to the ray $\{ z : \arg(w - \varphi(\zeta)) = A(\zeta) + \alpha \}$, as $z \rightarrow \zeta$. Thus $\partial \varphi(\Omega)$ has an inner tangent at $\varphi(\zeta)$ and $\varphi^{-1}$ is semi-conformal at $\varphi(\zeta)$. If $\varphi$ is a conformal map of the unit disk $\mathbb{D}$ onto a region $\Omega$ bounded by a Jordan curve $\Gamma$ and if $\Gamma$ has a tangent at $w \in \Gamma$, then by Carathéodory’s theorem and Lindelöf’s theorem (see e.g. Pommerenke [1975]), $\varphi$ is semi-conformal at $\zeta = \varphi^{-1}(w)$; moreover convergence $z \rightarrow \zeta$ is not restricted to cones. However in general, $\varphi$ can be semi-conformal at $\zeta \in \partial \mathbb{D}$ even though $\Gamma$ does not have a tangent at $\varphi(\zeta)$.

The next theorem, due to Ostrowski, gives geometric conditions on region $\Omega = \varphi(\mathbb{H})$ equivalent to the semi-conformality of $\varphi$ at $\zeta$, where $\mathbb{H}$ denotes the upper half plane $\{ z : \text{Im}z > 0 \}$. For convenience, we will state the case when $\zeta = 0$, $\varphi(0) = 0$ and when the limit $A(\zeta) = 0$ in (1); for the other cases, simply translate and rotate.

**Theorem 3.** (Ostrowski [1937]). Suppose $\Omega$ is a simply connected domain in $\mathbb{C}$. If the conformal map $\varphi$ of the upper half-plane $\mathbb{H}$ onto $\Omega$ is semi-conformal at 0 with $\varphi(0) = 0$, and if the limit in (1) is 0, then $\partial \Omega$ has an inner tangent at 0 with a vertical inner normal and

$$\lim_{x \in \partial \Omega, x \rightarrow 0} \frac{\text{dist}(x, \partial \Omega)}{|x|} = 0.$$
Conversely, if \( \partial \Omega \) has an inner tangent at \( 0 \) with a vertical inner normal and satisfies (2) then we can choose the conformal map \( \varphi \) of \( \mathbb{H} \) onto \( \Omega \) so that it is semi-conformal at \( 0 \) with non-tangential limit equal to 0 and so that the limit in (1) is 0.

\[
\begin{array}{c}
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\varphi(z) = \lim_{\Gamma_{\beta}(\zeta) \ni z \to \zeta} \varphi(z)
\end{array}
\end{array}
\]

for every \( \beta \in (0, \pi/2) \) and if

\[
\varphi'(\zeta) = \lim_{\Gamma_{\beta}(\zeta) \ni z \to \zeta} \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}
\]

exists for every \( \beta \in (0, \pi/2) \).

It is not hard to show that \( \varphi \) has an angular derivative \( \varphi'(\zeta) \) at \( \zeta \) if and only if \( \varphi' \) has non-tangential limit \( \varphi'(\zeta) \). See for example, Pommerenke [1975].

If \( \varphi \) has a non-zero angular derivative at \( \zeta \) then \( \varphi \) is semi-conformal at \( \zeta \), and \( \varphi^{-1} \) has a non-zero angular derivative at \( \varphi(\zeta) \). Preserving angles in a region \( \Omega \) is equivalent to having a non-zero derivative in \( \Omega \), however it is possible for \( \varphi \) to be semi-conformal at \( \zeta \in \partial \Omega \) and not have an angular derivative at \( \zeta \). A normal families argument does, however, give us some information about the modulus of the difference quotient if \( \varphi \) is semi-conformal at \( \zeta \). For if \( \partial \Omega \) has an inner tangent with vertical inner normal at \( 0 \in \partial \Omega \) and if \( \varphi \) is conformal on \( \Omega \) and semi-conformal at \( 0 \), consider the functions

\[
f_r(z) = \log \frac{\varphi(rz) - \varphi(0)}{rz} \cdot \frac{r i}{\varphi(r i) - \varphi(0)}
\]
Then $\text{Im} f_r$ converges to 0, as $r \to 0$, uniformly on compact subsets of the half-annulus

$$A = \{z : 1/2 < |z| < 2\} \cap \mathbb{H}.$$  

Since $\text{Re} f_r (i) = 0$,

$$\lim_{r \to 0} \left| \frac{\varphi(rz) - \varphi(0)}{rz} \cdot \frac{r^i}{\varphi(rz) - \varphi(0)} \right| = \lim_{r \to 0} e^{\text{Re} f_r} = 1$$

uniformly on compact subsets of $A$. Thus if $\varphi$ is semi-conformal at $\zeta$, then $\varphi$ has an angular derivative at 0 if and only if

$$\lim_{r \to 0} \left| \frac{\varphi(rz) - \varphi(0)}{rz} \right|$$

exists. This says that there may be stretching or compression, but only in the radial direction.

The **Angular Derivative Problem** is to give Euclidean geometric conditions on the boundary of a simply connected region $\Omega$ near $\zeta \in \partial \Omega$ which are equivalent to the existence of a non-zero angular derivative at $\zeta$ for the conformal map of $\Omega$ onto the half-plane or the disc. The existence of a non-zero angular derivative only depends on the geometry of $\Omega$ near $\zeta$ and does not depend on the choice of the conformal map.

**Definition 5.** If $\Omega$ is simply connected and if $0 \in \partial \Omega$, we say $\Omega$ has a **positive angular derivative** at 0 if there is a conformal map $\varphi$ of $\mathbb{H}$ onto $\Omega$ which has non-tangential limit $\varphi(0) = 0$ and which has angular derivative $\varphi'(0)$ with $0 < \varphi'(0) < \infty$.

By the remarks above, a characterization of simply connected domains with positive angular derivative at 0 would solve the Angular Derivative Problem.

The angular derivative problem has been converted into a problem about extremal length, which can be estimated in many cases by the geometry. If $E$ and $F$ are subsets of the closure, $\bar{\Omega}$, of $\Omega$, let $\Gamma$ denote the collection of locally rectifiable curves in $\Omega$ connecting $E$ to $F$. The **extremal distance** in $\Omega$ between $E$ and $F$ is defined to be

$$d_\Omega (E, F) = \sup_{\rho \geq 0} \frac{\left( \inf_{\gamma \in \Gamma} \int_{\gamma} \rho |dz| \right)^2}{\int_{\Omega} \rho^2 dA},$$

(5)

where $|dz|$ denotes arc length measure, $dA$ denotes Lebesgue area measure and where the supremum is taken over all non-negative Borel functions $\rho$ satisfying $0 < \int_{\Omega} \rho^2 dA < \infty$. (Such functions $\rho$ are called **metrics**.) For example, the extremal distance between the vertical ends of a rectangle, with sides parallel to the axes, is the ratio of the length to the height. Since extremal distance is conformally invariant, if $E_r = \{z : |z| = r\} \cap \mathbb{H}$, then for $t > s$, $d_\Omega (E_t, E_s) = \frac{1}{\pi} \log \frac{t}{s}$, where $\Omega$ is the half-annulus $\{z : s < |z| < t, \text{Im} z > 0\}$. The region $\Omega$ can also be taken to be the upper half plane $\mathbb{H}$. See Ahlfors [1973] for an introduction to extremal length.
Theorem 6. (Jenkins-Oikawa [1977], Rodin-Warschawski [1976]) Suppose \( \Omega \) is a simply connected domain containing the positive imaginary axis, with \( 0 \in \partial \Omega \). Let \( E_r \) denote the component of \( \{ z : |z| = r \} \cap \Omega \) containing \( ir \), for \( r > 0 \). Then \( \Omega \) has a positive angular derivative at 0 if and only if

(i) \( \Omega \) has an inner tangent at 0 with a vertical inner normal and

(ii) \( \lim_{s \to 0} d_{\Omega}(E_t, E_s) - \frac{1}{\pi} \log \frac{t}{s} = 0. \)

Condition (ii) says that the conformal map behaves like a constant multiple of \( z \), asymptotically. Note also that if (2) holds then by the argument used to establish (4), we can add the further restriction to condition (ii) that \( s = 2^{-n} \) and \( t = 2^{-m} \), \( n, m \in \{1, 2, 3, \ldots \} \) and the theorem still holds.

§2 Strip Domains.

It is convenient sometimes to transform both \( \Omega \) and \( \mathbb{H} \) to “strips”. The “standard strip”

\[
\mathbb{S} = \{ z : |\text{Im}z| < \pi/2 \}
\]

is mapped onto the upper half-plane \( \mathbb{H} \) by the function \( \tau(z) = ie^{-z} \). Then

\[
\psi(z) = \tau^{-1} \circ \varphi \circ \tau(z) = \frac{\pi i}{2} - \log \varphi(ie^{-z})
\]

is a conformal map of \( \mathbb{S} \) onto a region \( \Omega \) with

\[
\Omega = \{ ie^{-z} : z \in \Omega \}.
\]

Cones are replaced by half-strips

\[
S_\delta = \{ z : |\text{Im}z| < \pi/2 - \delta \text{ and Re}z > 0 \},
\]

for \( \delta \in (0, \pi/2) \). Thus \( \partial \Omega \) has an inner tangent at 0 with a vertical normal if and only if for every \( \delta \in (0, \pi/2) \) there is and \( x_\delta > 0 \) so that

\[
x_\delta + S_\delta \subset \Omega. \tag{7}
\]

The conformal map \( \varphi \) of \( \mathbb{H} \) onto \( \Omega \) has a non-tangential limit 0 at \( \zeta = 0 \) if and only if

\[
\text{Re}\psi(z) \to +\infty \text{ as } \text{Re}z \to +\infty, \quad z \in S_\delta \tag{8}
\]
for every \( \delta \in (0, \pi/2) \). In this case, \( \varphi \) is semi-conformal at 0 if and only if

\[
\lim_{z \to e^{i\delta} \atop z \in \Omega} \text{Im} \psi(z) - \text{Im} z \tag{9}
\]

exists for every \( \delta \in (0, \pi/2) \). Condition (2) in Ostrowski’s theorem is equivalent to

\[
\lim_{z \to \partial \Omega} \text{dist}(z, \partial \Omega) = 0. \tag{10}
\]

Likewise \( \varphi \) has non-tangential limit 0 and non-zero angular derivative at 0 if and only if

\[
\lim_{z \to e^{i\delta} \atop z \in \Omega} \psi(z) - z \tag{11}
\]

exists for every \( \delta \in (0, \pi/2) \).

It has become the custom to consider slightly more general circumstances, by removing the restriction that \( \tau \) be one-to-one.

**Definition 7.** If \( \Omega \) is simply connected and satisfies (7), let \( \psi \) be a conformal map of \( \Omega \) onto \( S \) which satisfies (8). The region \( \Omega \) is said to have an **angular derivative** (at \( +\infty \)) when (11) holds.

Clearly this definition does not depend on the choice of the map \( \psi \) and hence is a property of the region \( \Omega \). A geometric characterization of regions with angular derivative at \( +\infty \) would give a solution to the Angular Derivative Problem. Theorem 3 extends to this slightly more general context, if we use (9) as the definition of semi-conformality and replace (2) with (10). Likewise Theorem 6 extends by replacing (i) with (7) and replacing (ii) with

\[
\lim_{s,t \to +\infty} d_{\Omega}(F_s, F_t) - |t - s|/\pi = 0, \tag{12}
\]

where \( F_x \) denotes the component of \( \{z \in \Omega : \text{Re} z = x\} \) containing \( x \). This more general form of Theorem 6, using (7) and (12), is in Jenkins-Oikawa [1977] and Rodin-Warschawski [1976].

For strip domains, there are some classical estimates of extremal distance which motivate our results. If \( E \) and \( F \) are the vertical sides of the rectangle

\[
R = \{z : |\text{Im} z| < H, |\text{Re} z| < L\},
\]

then the extremal distance satisfies

\[
d_R(E, F) = L/H.
\]
By the conformal invariance of extremal distance, if \( \psi \) is a conformal map of a region \( U \) onto \( R \) with \( \psi(E') = E \) and \( \psi(F') = F \), then

\[
d_U(E', F') = L/H
\]

and the metric \( \rho = |\psi'| = |\nabla \text{Re} \psi| \) is extremal in the sense that the supremum in (5) is achieved with this metric.

Any metric \( \rho \) gives a lower bound for extremal distance. To motivate a choice of a metric below, we first consider the special case of a quadrilateral

\[
U = \{(x, y) : |y - m(x)| < \theta(x)/2, \ a < x < b, \}
\]
of width \( \theta(x) \) and mid-line \( y = m(x) \). Let \( \sigma_j = U \cap \{z : \text{Re} z = x_j\}, \ j = 0, \ldots, n \), where \( a = x_0 < x_1 < \ldots < x_n = b \). By the serial rule (see Ahlfors [1973]),

\[
d_U(\sigma_0, \sigma_n) \geq \sum_{j=1}^{n} d(\sigma_{j-1}, \sigma_j).
\]

The region between \( \sigma_{j-1} \) and \( \sigma_j \) is approximately at thin rectangle, if \( \Delta x = x_j - x_{j-1} \) is small. This rectangle can be mapped to a rectangle of height 1, centered on \( \mathbb{R} \) by the linear map

\[
\frac{z - im(x_j)}{\theta(x_j)}.
\]

Since the image rectangle has width \( \Delta x / \theta(x_j) \), this suggests the (non-analytic) map

\[
(x, y) \xrightarrow{\Phi} \left( \int_{a}^{b} \frac{dt}{\theta(t)}, \frac{y - m(x)}{\theta(x)} \right)
\]

of \( U \) onto a rectangle of height 1 and length \( \int_{a}^{b} \frac{1}{\theta(t)} dt \). The extremal metric for \( n_U(\sigma_0, \sigma_n) \) is given by \( \rho(z) = |\varphi'(z)| = |\nabla \text{Re} \varphi(z)| \) where \( \varphi \) is a conformal map of \( U \) onto a rectangle, so let

\[
\rho(z) = |\nabla \text{Re} \Phi| = \frac{1}{\theta(z)}
\]

where \( z = x + iy \). If \( \gamma \) is any curve connecting \( \sigma_0 \) to \( \sigma_n \) in \( U \), then

\[
\int_{\gamma} \rho(z)|dz| \geq \int_{a}^{b} \frac{1}{\theta(x)} dx.
\]

Moreover,

\[
\iint_{U} \rho^2 dy dx = \int_{a}^{b} \frac{1}{\theta(x)} dx
\]
and so
\[ d_U(\sigma_0, \sigma_n) \geq \int_a^b \frac{1}{\theta(x)} \, dx. \] (14)

More generally, suppose \( \Omega \) is a simply connected domain containing the real line \( \mathbb{R} \). Let \( F_x \) denote the component of \( \{ z \in \Omega : \text{Re} z = x \} \) containing \( x \). Fix \( s < t < \infty \), and define a metric \( \rho_A \) by
\[ \rho_A(x, y) = \begin{cases} \frac{1}{\theta(x)} & \text{if } s < x < t \text{ and } (x, y) \in F_x \\ 0 & \text{elsewhere in } \Omega. \end{cases} \]

Then by the argument above
\[ d_{\Omega}(F_s, F_t) \geq \int_s^t \frac{1}{\theta(x)} \, dx. \] (15)

This estimate is due to Ahlfors.

For regions of the form \( U = \{ (x, y) : |y - m(x)| < \theta(x)/2, \ a < x < b \} \) there is an upper bound for \( d_U(\sigma_0, \sigma_n) \) that complements (14). Notice that the extremal distance between the two horizontal sides of the rectangle \( R \), given above, is equal to the reciprocal of the extremal distance between the vertical sides, by a rotation. Any metric gives a lower bound for the extremal distance between the two components of \( \partial U \setminus (\sigma_0 \cup \sigma_n) \) and thus gives an upper bound for the extremal distance between \( \sigma_0 \) and \( \sigma_n \).

By (13), a natural choice for a metric is then
\[ \rho_B(x, y) = \left| \nabla \left( \frac{y - m(x)}{\theta(x)} \right) \right| = \sqrt{\frac{1}{\theta^2(x)} + \left( \frac{(y - m)\theta' + \theta m'}{\theta^2} \right)^2}. \] (16)

This metric was discovered by Beurling [1989].

If \( \gamma \) is any curve in \( U \) connecting the curve \( y = m(x) + \theta(x)/2 \) to the curve \( y = m(x) - \theta(x)/2 \), then
\[ \int_\gamma \rho_B \, dz \geq \left| \int_\gamma \nabla \left( \frac{y - m}{\theta} \right) \cdot dz \right| = 1. \] (17)

Furthermore,
\[ \int_U \rho_B^2 \, dA = \int_a^b \int_{m - \frac{\theta}{2}}^{m + \frac{\theta}{2}} \left\{ \frac{1}{\theta^2} + \left[ \frac{(y - m)\theta' + \theta m'}{\theta^2} \right]^2 \right\} \, dy \, dx \]
\[ = \int_a^b \frac{dx}{\theta(x)} + \int_a^b \frac{m'(x)^2 + \frac{1}{12} \theta(x)^2}{\theta(x)} \, dx. \]

Thus
\[ d_U(\sigma_0, \sigma_n) \leq \int_a^b \frac{dx}{\theta(x)} + \int_a^b \frac{m'(x)^2 + \frac{1}{12} \theta'(x)^2}{\theta(x)} \, dx. \]

For regions of the form \( \Omega = \{ x + iy : |y - m(x)| < \theta(x)/2 \} \), let \( F_x = \{ z \in \Omega : \text{Re} z = x \} \). Then for \( s < t \),
\[ d_\Omega(F_s, F_t) \leq \int_s^t \frac{dx}{\theta(x)} + \int_s^t \frac{m'(x)^2 + \frac{1}{12} \theta'(x)^2}{\theta(x)} \, dx. \] (18)
The inequality (18) was discovered by Warschawski [1942], using the older length-area method.

These estimates of Ahlfors and Beurling can be used to give necessary and sufficient geometric conditions for the existence of a positive angular derivative when \( \partial \Omega \) is sufficiently smooth. See Rodin-Warschawski [1976].

**Corollary 8.** Suppose \( \tilde{\Omega} \) is a simply connected domain given by
\[ \tilde{\Omega} = \{ x + iy : |y - m(x)| < \theta(x)/2 \} \]
where \( \lim_{x \to +\infty} m(x) = 0 \) and
\[ \int_0^\infty (m')^2 + (\theta')^2 \, dx < \infty. \] (19)

Then \( \tilde{\Omega} \) has an angular derivative at \( +\infty \) if and only if
\[ \int_0^\infty \left( \frac{1}{\theta(x)} - \frac{1}{\pi} \right) \, dx \]
exists.

**Proof.** If (20) holds then \( \lim_{s \to +\infty} \theta(s) = \pi \). Indeed, if \( \varepsilon > 0 \), then for \( s \) and \( t \) sufficiently large, with \( s < t < s + 1 \), by the Cauchy-Schwarz inequality and (19)
\[ |\theta(s) - \theta(t)| < \varepsilon. \]

If there is a sequence \( s_n \to +\infty \) with \( \theta(s_n) > \pi + 2\varepsilon \), then \( \theta(t) > \pi + \varepsilon \) for \( s_n < t < s_n + 1 \) and hence
\[ \int_{s_n}^{s_n+1} \left( \frac{1}{\theta(t)} - \frac{1}{\pi} \right) \, dt < -\frac{2\varepsilon}{\pi(\pi + 2\varepsilon)} \]
contradicting (20). A similar argument works if there is a sequence \( s_n \to +\infty \) with \( \theta(s_n) < \pi - 2\varepsilon \).
Thus (7) and (10) hold. As before, let \( F_x = \tilde{\Omega} \cap \{ z : \text{Re} z = x \} \). For \( s,t \) sufficiently large, by (15) and (18) we have the estimates
\[ \int_s^t \frac{1}{\theta(x)} \, dx \leq d_\Omega(F_r, F_s) \leq \int_s^t \frac{1}{\theta(x)} \, dx + \int_s^t \frac{m'(x)^2 + \frac{1}{12} \theta'(x)^2}{\theta(x)} \, dx. \]
Thus by the strip version of Theorem 6, \( \tilde{\Omega} \) has a angular derivative at \( +\infty \) if and only if

\[
\lim_{s,t \to +\infty} \int_s^t \left( \frac{1}{\theta(x)} - \frac{1}{\pi} \right) dx = 0. \tag{21}
\]

\( \square \)

\( \S 3 \) **Area Estimates.**

Condition (21) has a more geometric interpretation. If \( \tilde{\Omega} \) is a region containing the real line \( \mathbb{R} \), let \( F_r \) denote the component of \( \{ z : \text{Re} z = r \} \cap \tilde{\Omega} \) containing \( r \in \mathbb{R} \), and let

\[ \tilde{\Omega}' = \text{interior} \left( \bigcup_{r \in \mathbb{R}} F_r \right). \]

For any region \( U \), let

\[ U_{s,t} = U \cap \{ z : s < \text{Re} z < t \}. \]

Then the integral in (21) is related to the area measure of the difference between the standard strip \( S \) and the region \( \tilde{\Omega}' \) since

\[
\int_s^t \left( \frac{1}{\theta} - \frac{1}{\pi} \right) dx \geq \frac{1}{\pi^2} \int_s^t (\pi - \theta) dx
= \frac{1}{\pi^2} \left( \text{Area}(S \setminus \tilde{\Omega}')_{s,t} - \text{Area}(\tilde{\Omega}' \setminus S)_{s,t} \right). \tag{22}
\]

Thus if \( \tilde{\Omega} \) is a region containing \( \mathbb{R} \) then by (15)

\[
d_{\tilde{\Omega}}(F_s, F_t) - (t - s)/\pi \geq \frac{1}{\pi^2} \left( \text{Area}(S \setminus \tilde{\Omega}')_{s,t} - \text{Area}(\tilde{\Omega}' \setminus S)_{s,t} \right). \tag{23}
\]

If \( \theta(x) \to \pi \) as \( x \to +\infty \) (as was the case in Corollary 8) then the inequality (22) is almost an equality as \( s, t \to \infty \). For example if

\[ \text{Area}(S \setminus \tilde{\Omega}') + \text{Area}(\tilde{\Omega}' \setminus S) < \infty \]

then

\[
\lim_{s,t \to \infty} \int_s^t \left| \frac{1}{\theta(x)} - \frac{1}{\pi} \right| dx = 0.
\]

The above discussion becomes even more significant if we apply it not to \( \tilde{\Omega} \) but to a smaller region with Lipschitz boundary. If \( \tilde{\Omega} \) is a region containing \( \mathbb{R} \), let \( \mathcal{T}_M \) denote the collection of isosceles triangles contained in \( \tilde{\Omega} \) with base on \( \mathbb{R} \) and sides with slope \( \pm M \). Set

\[ \tilde{\Omega}_M = \bigcup \{ T : T \in \mathcal{T}_M \}. \]
An $M$-Lipschitz curve is the graph of a function $h$ which satisfies

$$|h(s) - h(t)| \leq M|s - t|$$

for all $s, t \in \mathbb{R}$. Then $\Omega_M$ is the largest subregion of $\Omega$ containing $\mathbb{R}$ which is bounded by $M$-Lipschitz curves.

**Theorem 9.** Suppose $\Omega$ is a simply connected domain with $\mathbb{R} \subset \Omega$ and

$$\partial \Omega \cap \{z : \operatorname{Re} z < 0\} = \partial \mathbb{S} \cap \{z : \operatorname{Re} z < 0\}. \quad (24)$$

Let $\Omega_M$ be the largest region bounded by $M$-Lipschitz curves contained in $\Omega$, as defined above.

(i) If $\operatorname{Area}(\mathbb{S} \setminus \Omega_M) < \infty$ then $\Omega$ has an angular derivative at $+\infty$ if and only if $\operatorname{Area}(\Omega_M \setminus \mathbb{S}) < \infty$.

(ii) Suppose $M > 8\pi$. If $\operatorname{Area}(\Omega_M \setminus \mathbb{S}) < \infty$, then $\Omega$ has an angular derivative at $+\infty$ if and only if $\operatorname{Area}(\mathbb{S} \setminus \Omega_M) < \infty$.

![Diagram of $\Omega$ and $\Omega_M$](image)

**Figure 2**

The statement (24) is inconsequential since the existence of an angular derivative at $+\infty$ is a local property of $\partial \Omega$ near $+\infty$.

Statement (i) was proved by Burdzy [1986] in the form given in Theorem 13(i) below using Brownian excursions. See also Carroll [1988] and Gardiner [1991]. Half of statement (i), in the form given above, was also proved by Rodin-Warschawski [1986] and the other half was also proved by Sastry [1995]. Statement (ii) and the proof of (i) below are new. Since there seems to be some uncertainty (see e.g. Sastry [1995]) about the exact relationship between Theorem 9(i) and Theorem 13(i), we will first prove Theorem 9, then derive Theorem 13 from Theorem 9. If $\tau(z) = ie^{-z}$ is
one-to-one on \( \tilde{\Omega} \) (see section 2) then one can derive Theorem 9 (for \( \tilde{\Omega} \)) from Theorem 13 in a similar manner, so long as \( 8\pi \) is replaced by a somewhat larger constant.

**Proof.** If \( \text{Area}(\mathcal{S} \setminus \tilde{\Omega}_M) < \infty \) and if \( \text{Area}(\tilde{\Omega}_M \setminus \mathcal{S}) < \infty \) then

\[
\lim_{x \to +\infty} \lim_{z \in \partial \tilde{\Omega}_M} \text{dist}(z, \partial \mathcal{S}) = 0,
\]

and hence (7) holds. Conversely, if \( \tilde{\Omega} \) has an angular derivative at \( +\infty \), then by Theorem 3, (25) holds. Thus we may suppose for the remainder of the proof that (25) holds. If \( F_s \) denotes the component of \( \tilde{\Omega} \cap \{ z : \text{Re} z = s \} \) containing \( s \in \mathbb{R} \), then by (25) and the strip version of Theorem 6, \( \tilde{\Omega} \) has an angular derivative at \( +\infty \) if and only if

\[
\lim_{x \to +\infty} \lim_{s \to t, s < t} d_{\tilde{\Omega}}(F_s, F_t) - (t - s)/\pi = 0.
\]

To prove Theorem 9, we will obtain lower and upper bounds for the left side of (26) in Lemma 10 and Lemma 12 below.

Suppose that \( \partial \tilde{\Omega}_M \) is given by the two curves \( y = h_1(x) + \pi/2 \) and \( y = -h_2(x) - \pi/2 \), \( -\infty < x < +\infty \), where \( h_1 > -\pi/2 \) and \( h_2 > -\pi/2 \). By (25), \( \lim_{x \to +\infty} h_j(x) = 0 \). As before, for any region \( U \), let

\[
U_{s,t} = U \cap \{ z : s < \text{Re} z < t \}.
\]

**Lemma 10.** (Sastry [1995]) Suppose \( |h_j(x)| \leq \pi M^2 \) for \( s < x < t \). Then

\[
d_{\tilde{\Omega}}(F_s, F_t) - (t-s)/\pi \leq \frac{4M^2+2}{\pi} \text{Area}(\mathcal{S} \setminus \tilde{\Omega}_M)_{s,t} - \frac{1}{4M^2+2} \pi \text{Area}(\tilde{\Omega}_M \setminus \mathcal{S})_{s,t}.
\]

Sastry [1995] proved this lemma with different constants and used a piecewise constant metric. We will use a metric of the form \( \rho = |\nabla u| \), since it is much easier to estimate lengths as we have seen for example in (17).

**Proof.** Since \( d_{\tilde{\Omega}}(F_s, F_t) \leq d_{\tilde{\Omega}_M}(F_s, F_t) = d_{(\tilde{\Omega}_M)_{s,t}}(F_s, F_t) \), without loss of generality, we may suppose (for notational convenience) that \( \tilde{\Omega} = (\tilde{\Omega}_M)_{s,t} \). Set \( \lambda = 1/(2M^2) \) and define

\[
U_1 = \{ (x,y) \in \tilde{\Omega} : h_1(x) > 0 \text{ and } -\lambda h_1 + \pi/2 < y < h_1 + \pi/2 \}
\]

\[
U_2 = \{ (x,y) \in \tilde{\Omega} : h_1(x) < 0 \text{ and } 2h_1 + \pi/2 < y < h_1 + \pi/2 \}
\]

\[
U_3 = \{ (x,y) \in \tilde{\Omega} : h_2(x) > 0 \text{ and } -h_2 - \pi/2 < y < \lambda h_2 - \pi/2 \}
\]

\[
U_4 = \{ (x,y) \in \tilde{\Omega} : h_2(x) < 0 \text{ and } -h_2 - \pi/2 < y < -2h_2 - \pi/2 \}
\]

\[
U^* = \tilde{\Omega} \setminus \bigcup_{j=1}^4 U_j.
\]
Consider the continuous map \((x, y) \to (x, u(x, y))\) of \(\Omega\) onto \(\mathbb{S}\) which fixes \(U^*\) and is linear in \(y\) on \(\bar{\Omega} \setminus U^*\). Thus

\[
u(x, y) = \begin{cases} 
  y & \text{for } (x, y) \in U^* \\
  (y - (h_1 + \pi/2))/(2M^2 + 1) + \pi/2 & \text{for } (x, y) \in U_1 \\
  2(y - (h_1 + \pi/2)) + \pi/2 & \text{for } (x, y) \in U_2 \\
  (y + h_2 + \pi/2)/(2M^2 + 1) - \pi/2 & \text{for } (x, y) \in U_3 \\
  2(y + h_2 + \pi/2) - \pi/2 & \text{for } (x, y) \in U_4.
\end{cases}
\]

Set \(\rho = |\nabla u|\) on \(\bar{\Omega}\). This metric is similar to Beurling's metric in section 2, except that we fix the region \(U^*\). Note that

\[
|\nabla u|^2 \leq \begin{cases} 
  1 & \text{on } U^* \\
  4(1 + M^2) & \text{on } U_2 \cup U_4 \\
  (1 + M^2)/(2M^2 + 1)^2 & \text{on } U_1 \cup U_3
\end{cases}.
\]

Thus

\[
\int_{\bar{\Omega}} |\nabla u|^2 dydx - \int_{S_{s,t}} dydx \leq 4(1 + M^2)\text{Area}(U_2 \cup U_4) + \frac{1 + M^2}{(2M^2 + 1)^2}\text{Area}(U_1 \cup U_3) - \frac{1}{2M^2}\text{Area}(\bar{\Omega} \setminus S_{s,t}) - 2\text{Area}(S_{s,t} \setminus \Omega) = (4M^2 + 2)\text{Area}(S_{s,t} \setminus \Omega) - \frac{1}{4M^2 + 2}\text{Area}(\bar{\Omega} \setminus S_{s,t}).
\]

If \(\gamma\) is a curve connecting \(\{y = h_1(x) - \pi/2\}\) to \(\{y = -h_2(x) - \pi/2\}\) and contained in \(\bar{\Omega}\), then

\[
\int_{\gamma} |\nabla u||dz| = \left| \int_{\gamma} \nabla u \cdot dz \right| = \pi.
\]
Thus
\[
d_{\Omega}(F_s, F_t) \leq \frac{\int_{\Omega} |\nabla u|^2 dydx}{(\inf_{\gamma} \int_{\gamma} |\nabla u| |dz|)^2}
\leq \frac{1}{\pi^2} \left( \pi(t - s) + (4M^2 + 2)\text{Area}(S_{s,t} \setminus \Omega) - \frac{1}{4M^2 + 2}\text{Area}(\Omega \setminus S_{s,t}) \right).
\]

\[\square\]

The next step is to derive a lower bound for extremal distance. Write \( \partial \Omega_M - \partial \Omega = \cup_j \sigma_j \) where each \( \sigma_j \) is an open arc with endpoints \( z^l_j, z^r_j \in \partial \Omega \) where \( \text{Re} z^l_j < \text{Re} z^r_j \). For \( v \in \mathbb{C} \setminus \mathbb{R} \), let \( T_v \) denote the triangle in \( T_M \) with vertex \( v \). There is a unique \( v_j \) with \( \text{Re} z^l_j \leq \text{Re} v_j \leq \text{Re} z^r_j \) so that \( z^l_j, z^r_j \in \partial T_{v_j} \). Then \( \sigma_j \subset \partial T_{v_j} \) and \( \sigma_j \) consists of at most two line segments. If \( |\text{Im} z^l_j| \leq |\text{Im} z^r_j| \), let \( B_j = \{ z : |z - z^l_j| < \text{Re} z^r_j - \text{Re} z^l_j \} \). Otherwise let \( B_j = \{ z : |z - z^r_j| < \text{Re} z^r_j - \text{Re} z^l_j \} \).

![Diagram](image.png)

\text{Figure 4}

**Lemma 11.**
\[
\text{Area}(B_j) \leq \frac{8\pi}{M} \int_{z \in \sigma_j} \left| |\text{Im} z| - \frac{\pi}{2} \right| dx, \quad z = x + iy.
\]

**Proof.** Suppose \( f \) is defined and continuous on \([0, 1]\) with
\[
f'(x) = \begin{cases} 1 & 0 \leq x \leq a \\ -1 & a \leq x \leq 1. \end{cases}
\]

Then by elementary calculus
\[
\int_0^1 |f(x)| dx \geq \frac{1}{8}.
\]

If \( \text{Im} z > 0 \) on \( \sigma_j \), the map
\[
(x, y) \rightarrow \left( \frac{x - \text{Re} z^l_j}{\text{Re} z^r_j - \text{Re} z^l_j}, \frac{y - \pi/2}{M(\text{Re} z^r_j - \text{Re} z^l_j)} \right)
\]

14
transforms \( \sigma_j \) into the graph of one such \( f \). Thus

\[
\int_{z \in \sigma_j} |\text{Im} z - \frac{\pi}{2}| \, dx \geq \frac{M(\text{Re} z_j^\Gamma - \text{Re} z_j^\Phi)^2}{8} = \frac{M}{8\pi} \text{Area}(B_j).
\]

The inequality of the lemma for \( \text{Im} z < 0 \) follows by a reflection about \( \Re \).

The quantity

\[
\int_{z \in \sigma_j} \left| |\text{Im} z| - \frac{\pi}{2} \right| \, dx
\]

is the total area “between” \( \sigma_j \) and \( \partial \mathcal{S} \).

The next lemma gives a lower bound for the extremal distance in \( \tilde{\Omega} \) using the geometry of \( \tilde{\Omega}_M \). When \( \tilde{\Omega} = \tilde{\Omega}_M \), it follows immediately from (23).

**Lemma 12.** If \( \varepsilon > 0 \) there is an \( s_0 < \infty \) so that for \( s < t < \infty \),

\[
d_{\tilde{\Omega}}(F_s, F_t) - (t - s) / \pi \geq \frac{M - s\pi}{M\pi} \text{Area}(\mathcal{S} \setminus \tilde{\Omega}_M)_{s,t} - \frac{M + s\pi}{M\pi} \text{Area}(\tilde{\Omega}_M \setminus \mathcal{S})_{s,t} - \varepsilon.
\]

**Proof.** If \( 0 \leq t - s \leq \varepsilon \), then by (25) \( \text{Area}(\tilde{\Omega}_M \setminus \mathcal{S})_{s,t} \to 0 \) and \( \text{Area}(\mathcal{S} \setminus \tilde{\Omega}_M)_{s,t} \to 0 \) as \( s, t \to \infty \), and the inequality follows. Now suppose that \( t - s \geq \varepsilon \). Let \( \rho \) be the metric on \( \tilde{\Omega} \) given by

\[
\rho = \begin{cases} 
1 & \text{for } z \in \tilde{\Omega}_M \cup \bigcup_j B_j \\
0 & \text{elsewhere on } \tilde{\Omega}.
\end{cases}
\]

The metric \( \rho \) will provide the lower bound for the extremal distance. First we compute the \( \rho \)-area of \( (\tilde{\Omega}_M)_{s,t} \). By Lemma 11

\[
\int_{(\tilde{\Omega}_M)_{s,t}} \rho^2 \, dx \, dy - \int_{\mathcal{S}_{s,t}} \, dx \, dy \leq \text{Area}(\tilde{\Omega}_M \setminus \mathcal{S})_{s,t} - \text{Area}(\mathcal{S} \setminus \tilde{\Omega}_M)_{s,t} + \sum_j \text{Area}(B_j)
\]

\[
\leq (1 + \frac{8\pi}{M}) \text{Area}(\tilde{\Omega}_M \setminus \mathcal{S})_{s,t} - (1 - \frac{8\pi}{M}) \text{Area}(\mathcal{S} \setminus \tilde{\Omega}_M)_{s,t}.
\]

Now suppose \( \gamma \) is a curve in \( \tilde{\Omega} \) connecting \( F_s \) to \( F_t \), with \( s < t \). Note that

\[
\text{Re} z_j^\Gamma - \text{Re} z_j^\Phi \leq \frac{4}{M} \max_{z \in \sigma_j} \left| |\text{Im} z| - \pi/2 \right| \to 0
\]

as \( \sigma_j \to +\infty \), by (25). Thus by deleting an initial and terminal portion of \( \gamma \), if necessary, we may suppose that \( \gamma \) begins and ends in \( \tilde{\Omega}_M \cup \bigcup_j B_j \) and that if \( \gamma \) meets \( \sigma_j \), then \( s < \text{Re} z_j^\Gamma < \text{Re} z_j^\Phi < t \).
Thus we can write \( \gamma = \bigcup_k \gamma_k \) where \( \{ \gamma_k \} \) are disjoint subarcs of \( \gamma \) such that either \( \gamma_k \subset \bar{\Omega}_M \cup \bigcup_j B_j \), or \( \gamma_k \) meets some \( \sigma_j \) and connects \( F_{\text{Re}z_j} \) to \( F_{\text{Re}z_j}^t \) with \( s < \text{Re}z_j < \text{Re}z_j^t < t \). If \( \gamma_k \subset \bar{\Omega}_M \cup \bigcup_j B_j \), then \( \int_{\gamma_k} \rho|dz| \) is at least equal to the change in Rez along \( \gamma_k \). If \( \gamma_k \) connects \( F_{\text{Re}z_j} \) to \( F_{\text{Re}z_j}^t \) and intersects \( \sigma_j \), then
\[
\text{length}(\gamma_k \cap (\bar{\Omega}_M \cup B_j)) \geq \text{Re}z_j^t - \text{Re}z_j^t.
\]
Indeed, if \( z_j^t \) is the center of \( B_j \) and if \( \zeta \in \gamma_k \cap \sigma_j \), then the shortest curve from \( F_{\text{Re}z_j} \) to \( \zeta \) in \( \bar{\Omega}_M \cup B_j \) is the straight line from \( z_j^t \) to \( \zeta \), since \( |\text{Im}\zeta| \geq |\text{Im}z_j^t| \). Since \( \gamma_k \) must intersect \( \partial B_j \), it must have length at least the radius of \( B_j \), namely \( \text{Re}z_j^t - \text{Re}z_j^t \). A similar argument works if \( z_j^t \) is the center of \( B_j \).

Together with (28) this implies, for \( s, t \) sufficiently large, that
\[
\int_{\gamma} \rho|dz| \geq t - s - \varepsilon.
\]
We conclude by (27) that for \( t - s \geq \varepsilon \),
\[
d_{\bar{\Omega}}(F_s, F_t) \geq \frac{(t - s - \varepsilon)^2}{\pi(t - s) + (1 + 8\pi/M)\text{Area}(\Omega_M \setminus S)_{s,t} - (1 - 8\pi/M)\text{Area}(S \setminus \bar{\Omega}_M)_{s,t}}.
\]
Since \( \partial \bar{\Omega}_M \) is Lipschitz and (25) holds, we have both \( \text{Area}(\bar{\Omega}_M \setminus S)_{s,t} \to 0 \) and \( \text{Area}(S \setminus \bar{\Omega}_M)_{s,t} \to 0 \) as \( s, t \to \infty \). Hence for \( s, t \) sufficiently large with \( t - s \geq \varepsilon \),
\[
d_{\bar{\Omega}}(F_s, F_t) - (t - s)/\pi \geq \frac{M-8\pi}{M\pi^2}\text{Area}(S \setminus \bar{\Omega}_M)_{s,t} - \frac{M+8\pi}{M\pi^2}\text{Area}(\bar{\Omega}_M \setminus S)_{s,t} - \varepsilon.
\]

\[\Box\]

We can now conclude the proof of Theorem 9. The condition \( \text{Area}(S \setminus \bar{\Omega}_M) < \infty \) is equivalent to \( \text{Area}(S \setminus \bar{\Omega}_M)_{s,t} \to 0 \) as \( s, t \to +\infty \). Likewise, the condition \( \text{Area}(\bar{\Omega}_M \setminus S) < \infty \) is equivalent to
Area(\(\bar{\Omega}_M \setminus \mathcal{S}\))_{s,t} \to 0 as \(s,t \to +\infty\). Thus if Area(\(\mathcal{S} \setminus \bar{\Omega}_M\)) < \(\infty\) and Area(\(\bar{\Omega}_M \setminus \mathcal{S}\)) < \(\infty\), by Lemma 10 and Lemma 12,

\[
\lim_{s,t \to +\infty} d_{\Omega}(F_s, F_t) - (t - s)/\pi = 0,
\]

and hence by the discussion at the beginning of the proof of the Theorem 9, \(\bar{\Omega}\) has an angular derivative at +\(\infty\).

If Area(\(\mathcal{S} \setminus \Omega_M\)) < \(\infty\) and \(\bar{\Omega}\) has an angular derivative at +\(\infty\), then by (26) and Lemma 10, Area(\(\bar{\Omega}_M \setminus \mathcal{S}\))_{s,t} \to 0 as \(s,t \to +\infty\), proving statement (i) of Theorem 9.

If \(M > 8\pi\) and if Area(\(\bar{\Omega}_M \setminus \mathcal{S}\)) < \(\infty\) then by (26) and Lemma 12, Area(\(\mathcal{S} \setminus \bar{\Omega}_M\))_{s,t} \to 0 as \(s,t \to +\infty\), completing the proof of Theorem 9.

Theorem 9 does not solve the Angular Derivative Problem, even for regions bounded by Lipschitz curves. For example, if \(\bar{\Omega} = \{(x,y) : |y - m(x)| < 1/2\}\) is a strip of constant width with \(|m'(x)| \leq 1\) and

\[
\int_0^\infty |m(x)|\,dx = \infty
\]

then \(\bar{\Omega}\) is bounded by two Lipschitz curves, but Area(\(\mathcal{S} \setminus \bar{\Omega}\)) = Area(\(\bar{\Omega} \setminus \mathcal{S}\)) = \(\infty\). Thus Theorem 9 does not apply. However if we also have \(m \to 0\) and

\[
\int_0^\infty |m'(x)|^2\,dx < \infty,
\]

then by Corollary 8, \(\bar{\Omega}\) has an angular derivative at +\(\infty\).

Theorem 9 can be restated using Lipschitz majorants for the boundary of \(\Omega\) with 0 \(\in\partial\Omega\). Suppose \(\Omega\) is a plane domain. Let

\[
\Gamma_M = \{z : \text{Im}z > M\text{Re}z\}
\]

be the cone with vertex at 0 and with sides of slope \(\pm M\), and let

\[
\Omega^M = \bigcup \{\zeta + \Gamma_M : \zeta + \Gamma_M \subset \Omega\}.
\]

If \(\Omega^M\) is not empty, it is bounded by the graph of an \(M\)-Lipschitz function \(h_M\). Set \(h_M = +\infty\) if \(\Omega^M = \emptyset\). We will call \(h_M\) the smallest \(M\)-Lipschitz majorant of \(\partial\Omega\), for if the graph of an \(M\)-Lipschitz function \(h\) is contained in \(\Omega\) then \(h \geq h_M\).
**Theorem 13.** Let \( \Omega \) be a simply connected domain with \( 0 \in \partial \Omega \) and let \( h_M \) denote the smallest \( M \)-Lipschitz majorant of \( \partial \Omega \).

(i) (Burdzy [1986]). Suppose
\[
\int_{-1}^{1} \chi_{h_M > 0} \frac{h_M(x)}{x^2} \, dx < \infty.
\]
Then \( \Omega \) has a positive angular derivative at 0 if and only if
\[
\int_{-1}^{1} \chi_{h_M < 0} \frac{h_M(x)}{x^2} \, dx > -\infty.
\]

(ii) There is an \( M_0 < \infty \) so that if \( M > M_0 \) and if \( h_M \) satisfies (30) then \( \Omega \) has a positive angular derivative at 0 if and only if \( h_M \) satisfies (29).

Theorem 13(ii) answers a question in Burdzy [1986] and Burdzy [1987] (Problem 11.7, page 164). In the next section we will give an example (suggested by Burdzy) where Theorem 9 (ii) and Theorem 13 (ii) fail for small \( M_0 \).

**Proof.** Since the existence of an angular derivative is a local property depending only on \( \Omega \) near 0, we may suppose that \( \partial \Omega \cap \{z : |z| > 1\} = \{z \in \mathbb{R} : |z| > 1\} \). If \( h_M \) satisfies (29) and (30) then \( h_M(x)/x \to 0 \) as \( x \to 0 \). For if \( h_M(x_n) > |x_n| > 0 \) then \( h_M(x) > |x_n/2| \) on an interval centered at \( x_n \) of length \( |x_n|/M \). This contradicts the integrability condition (29) if \( x_n \to 0 \). A similar argument holds if \( h_M(x_n) < -|x_n| < 0 \) using condition (30). Thus \( \partial \Omega \) has an inner tangent at 0 with a vertical inner normal.

Conversely, if \( \Omega \) has a positive angular derivative at 0 then by Theorem 3, \( \partial \Omega \) has an inner tangent with a vertical inner normal and (2) holds. This implies \( h_M(x)/x \to 0 \) as \( x \to 0 \). Thus we may suppose, for the remainder of the proof that \( h_M(x)/x \to 0 \) as \( x \to 0 \).

Let \( \gamma_M \) denote the graph of \( h_M \). If \( N > M \), then \( \gamma_M \cap \partial \Omega \subset \gamma_N \cap \partial \Omega \) and \( h_N \leq h_M \). Write
\[
\gamma_M \setminus \partial \Omega = \bigcup_j \sigma_j.
\]

As in the proof of Theorem 9, each \( \sigma_j \) consists of two intervals, the leftmost with slope \(-M\) and the rightmost with slope \(M\), and endpoints \( z_j^l, z_j^r \in \partial \Omega \). For \( N > M \) there is a unique cone \( \zeta_j + \Gamma_N \) with \( z_j^l, z_j^r \in \zeta_j + \partial \Gamma_N \). Moreover the graph of \( h_N(x) \) lies above \( \zeta_j + \partial \Gamma_N \) and below \( \sigma_j \), for \( \text{Re} z_j^l \leq x \leq \text{Re} z_j^r \). Thus \( \max|h_N| \leq C \max|h_M| \), for \( \text{Re} z_j^l \leq x \leq \text{Re} z_j^r \), where \( C \) is a constant depending only on \( M \) and \( N \). In particular, for \( N > M \), \( h_N(x)/x \to 0 \) as \( x \to 0 \).
Suppose first that (29) holds and that \( \Omega \) has a positive angular derivative at 0. Since
\[
\lim_{x \to 0} h_M(x)/x = 0,
\]
the graph of \( h_M \), for small \( x \), is transformed by the map
\[
\varphi(z) = \pi i/2 - \log z
\]
to two \( M' \) Lipschitz curves, where \( M' \) is a constant depending only on \( M \). Set \( \tilde{\Omega} = \varphi(\Omega) \). These curves must then be contained in the region \( \tilde{\Omega}_{M'} \supset \tilde{\Omega}_M \). Since \( h_M(x)/x \to 0 \), condition (29) implies
\[
\lim_{s,t \to +\infty} \text{Area}(S_{s,t} \setminus W) = 0,
\]
where \( W \) is the image of the region \( \{x + iy : |x| < 1, y > \max(h_m(x), 0)\} \) by the map \( \varphi \). Since \( W \subset \tilde{\Omega}_{M'} \), (29) implies
\[
\text{Area}(S \setminus \tilde{\Omega}_{M'}) < \infty.
\]
By Theorem 9 (i), \( \text{Area}(\tilde{\Omega}_{M'} \setminus S) < \infty \). Since \( \varphi^{-1}(\tilde{\Omega}_{M'}) \) contains the graph of \( h_M \), and since \( h_M(x)/x \to 0 \), as \( x \to 0 \), we must have (30).

There is a constant \( M_0 \) so that the region above the graph of \( h_M \), \( M \geq M_0 \), contains \( \varphi^{-1}(\tilde{\Omega}_N) \) where \( N = 8\pi + 1 \). Suppose now that \( M \geq M_0 \), that condition (30) holds and that \( \Omega \) has a positive angular derivative at 0. As above this implies \( \text{Area}(\tilde{\Omega}_N \setminus S) < \infty \). By Theorem 9, \( \text{Area}(S \setminus \tilde{\Omega}_N) < \infty \). As above, this implies (29).

Finally suppose that both (29) and (30) hold.

**Lemma 14.** Suppose \( 0 < M < N < \infty \) and suppose \( f \) and \( g \) are continuous on \([a,b]\) with \( f(a) = g(a) \), \( f(b) = g(b) \),
\[
f'(x) = \begin{cases} -M & \text{for } a \leq x \leq c \\ M & \text{for } c \leq x \leq b \end{cases}
\]
and
\[
g'(x) = \begin{cases} -N & \text{for } a \leq x \leq d \\ N & \text{for } d \leq x \leq b \end{cases}
\]
There is a constant \( C \) depending only on \( M \) and \( N \) so that if \( g \leq h \leq f \) on \([a,b]\), then
\[
\int_a^b |h(x)|dx \leq C \int_a^b |f(x)|dx.
\]

Before proving the lemma, let us use it to complete the proof of Theorem 13. As above, let \( \gamma_M \) and \( \gamma_N \) denote the graphs of \( h_M \) and \( h_N \), respectively, where \( N > M \). Write \( \gamma_M = (\gamma_M \cap \partial \Omega) \cup \sigma_j \)
where each $\sigma_j$ consists of two line segments of slope $\pm M$ on the interval $I_j = [a_j, b_j]$. As noted above, the endpoints of $\sigma_j$ belong to $\partial \Omega$.

By the lemma with $a = a_j, b = b_j, f = h_M$ and $h = h_N$ we have

$$
\int_{I_j} |h_N(x)| \, dx \leq C \int_{I_j} |h_M(x)| \, dx.
$$

Since $h_M(x)/x \to 0$ as $x \to 0$, we must have $(b_j - a_j)/a_j \to 0$ as $b_j \to 0$, and hence

$$
\int_{I_j} \frac{|h_N(x)|}{x^2} \, dx \leq \left( \frac{b_j}{a_j} \right)^2 \int_{I_j} \frac{|h_N(x)|}{b^2} \, dx \leq C \left( \frac{b_j}{a_j} \right)^2 \int_{I_j} \frac{|h_M(x)|}{x^2} \, dx.
$$

Since $\gamma_M \cap \partial \Omega \subset \gamma_N \cap \partial \Omega$, we have $h_N = h_M$ on $\mathbb{R} \setminus \cup_j I_j$. Thus

$$
\int_{-1}^1 \frac{|h_N(x)|}{x^2} \, dx \leq C \int_{-1}^1 \frac{|h_M(x)|}{x^2} \, dx.
$$

In other words, (29) and (30) hold with $M$ replaced by $N$, for all $N > M$.

Choose $M'$ and $N > M'$ so that $\varphi^{-1}(\overline{\Omega}_{M'})$ contains the graph of $h_M$ and lies above the graph of $h_N$. By (29) $\text{Area}(\mathbb{S} \setminus \overline{\Omega}_{M'}) < \infty$ and by (30) with $M$ replaced by $N$, $\text{Area}(\overline{\Omega}_{M'} \setminus \mathbb{S}) < \infty$. By Theorem 9, $\Omega$ has a positive angular derivative at 0.

\[ \Box \]

**Proof of Lemma 14.** If $\max |f| \geq 2M(b-a)$ then $\min |f| \geq \max |f| - M(b-a) \geq \frac{1}{2} \max |f|$, and hence

$$
\int_a^b |f(x)| \, dx \geq \frac{1}{2} \max |f|(b-a).
$$

If $\max |f| < 2M(b-a)$ then as in the proof of Lemma 11,

$$
\int_a^b |f(x)| \, dx \geq \frac{1}{8} M(b-a)^2 \geq \frac{1}{16} \max |f|(b-a).
$$

Since $\max |f| \leq \max |g| \leq 4\frac{N}{M} \max |f|$ and $g \leq h \leq f$, we have

$$
|h| \leq 4\frac{N}{M} \max |f|.
$$

We conclude that

$$
\int_a^b |h| \, dx \leq 4\frac{N}{M} \max |f|(b-a) \leq \frac{64N}{M} \int_a^b |f(x)| \, dx.
$$

\[ \Box \]

**Corollary 15.** Let $\Omega$ be a simply connected domain containing the upper half plane $\mathbb{H}$, with $0 \in \partial \Omega$ and let $h_M$ be the smallest $M$-Lipschitz majorant of $\partial \Omega$. Then $\Omega$ has a positive angular derivative at 0 if and only if

$$
\int_{-1}^1 \frac{h_M(x)}{x^2} \, dx > -\infty.
$$

20
This Corollary follows immediately from Theorem 13(i). See also Rodin-Warschawski [1977] for other equivalent conditions.

**Corollary 16.** Let \( \Omega \) be a simply connected domain contained in the upper half plane \( \mathbb{H} \) with \( 0 \in \partial \Omega \). Suppose the smallest \( M \)-Lipschitz majorant of \( \partial \Omega, h_M \), is not identically \(+\infty\). Then \( \Omega \) has a positive angular derivative at \( 0 \) if and only if

\[
\int_{-1}^{1} \frac{h_M(x)}{x^2} \, dx < \infty.
\]

This Corollary follows immediately from Theorem 13(ii) and Proposition 17 below. See also Rodin-Warschawski [1977] for other equivalent conditions. Each of these Corollaries has a version for strip regions that follows immediately from Theorem 9.

§4 Further Results.

Burdzy (private communication) suggested the following example where Theorem 9 (ii) fails for small \( M \). A similar example fails for the half plane version, Theorem 13. Suppose \( 0 < \varepsilon_n \to 0 \), and \( \sum \varepsilon_n^2 = \infty \). For \( \delta > 0 \), let

\[
h_n(x) = \begin{cases} 
 x & \text{for } 0 \leq x \leq \varepsilon_n \\
 2\varepsilon_n - x & \text{for } \varepsilon_n \leq x \leq 2\varepsilon_n + \delta \\
 x - 2(\varepsilon_n + \delta) & \text{for } 2\varepsilon_n + \delta \leq x \leq 2(\varepsilon_n + \delta) \\
 0 & \text{elsewhere on } \mathbb{R}.
\end{cases}
\]

Then the curve \( y = h_n(x) \) is 1-Lipschitz. Set

\[
\overline{\Omega}_n = \{(x, y) : h_n(x) - \pi/2 < y < \pi/2\}
\]

and set \( s = -1 \) and \( t = +1 \). Using the constant metric \( \rho \equiv 1 \), we obtain the lower bound

\[
d_{\overline{\Omega}_n}(F_s, F_t) - (t - s)/\pi \geq \frac{(\varepsilon_n^2 - \delta^2)2}{\pi(2\pi - \varepsilon_n^2)}.
\]

By Lemma 10, since \( \partial \Omega \) is 1-Lipschitz,

\[
d_{\overline{\Omega}_n}(F_s, F_t) - (t - s)/\pi \leq \frac{6\varepsilon_n^2}{\pi^2} - \frac{1}{6\pi^2}\delta^2.
\]

Thus we can choose \( \delta = \delta_n \) with \( \varepsilon_n \leq \delta_n \leq 6\varepsilon_n \) so that for \( s = -1 \) and \( t = +1 \),

\[
d_{\overline{\Omega}_n}(F_s, F_t) = (t - s)/\pi.
\]
We can then choose the conformal map $\varphi_n$ of $\tilde{\Omega}_n$ onto $\mathbb{S}$ so that $\varphi_n$ converges to the identity map, uniformly on 

$$\left(\mathbb{S} \setminus \{|z + \pi i/2| < 1/K\}\right) \cap \{|z| < K\}$$

for all $K > 0$.

Define 

$$h(x) = h_{n_j}(x - 2n_j)$$

for $2n_j - 1 < x < 2n_j + 1$, $j = 1, 2, \ldots$, set $h(x) = 0$ elsewhere in $\mathbb{R}$ and set 

$$\tilde{\Omega} = \{(x, y) : h(x) - \pi/2 < y < \pi/2\}.$$ 

Then we can choose $n_j \to \infty$ so that $\tilde{\Omega}$ has an angular derivative at $+\infty$. Note that for $M \leq 1/13$, 

$$\tilde{\Omega}_M \subset \mathbb{S}$$

and 

$$\text{Area}(\mathbb{S} \setminus \tilde{\Omega}_M) = \sum \varepsilon_n^2 = +\infty.$$ 

Thus Theorem 9 (ii) cannot hold with $M = 1/13$.

Lemma 14 can be improved to give the next Proposition.

**Proposition 17.** Let $\Omega$ be a simply connected domain with $0 \in \partial \Omega$ and for $M < \infty$, let $h_M$ denote the smallest $M$-Lipschitz majorant of $\partial \Omega$. Then 

$$\int_{-1}^{1} \frac{|h_M(x)|}{x^2} dx < \infty$$

holds for some $M > 0$ if and only if it holds for all $M > 0$.

Proposition 17 follows from the next Lemma as in the proof of Theorem 9. A similar statement holds for strip regions using areas.

**Lemma 18.** Suppose $0 < M < N < \infty$ and suppose $f$ and $g$ are continuous on $[a, b]$ with 

$$f(a) = g(a), f(b) = g(b),$$

$$f'(x) = \begin{cases} -M & \text{for } a \leq x \leq c \\ M & \text{for } c \leq x \leq b \end{cases}$$

and 

$$g'(x) = \begin{cases} -N & \text{for } a \leq x \leq d \\ N & \text{for } d \leq x \leq b \end{cases}$$

with
There is a constant \( C \) depending only on \( M \) and \( N \) so that if \( g \leq h \leq f \) on \([a, b]\), then
\[
\frac{1}{C} \int_a^b |f(x)| \, dx \leq \int_a^b |h(x)| \, dx \leq C \int_a^b |f(x)| \, dx.
\]

**Proof.** The upper bound follows from Lemma 14. To prove the lower bound, first suppose that \( \min g \leq 0 \). Write
\[
\{x \in (a, b) : g(x) > 0\} = (a, a_1) \cup (a_4, b),
\]
and
\[
\{x \in (a, b) : f(x) < 0\} = (a_2, a_3),
\]
where \( a \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq b \). Set
\[
h_1(x) = \begin{cases} 
g(x) & \text{if } g(x) > 0 \\
0 & \text{if } g(x) < 0 < f(x) \\
f(x) & \text{if } f(x) < 0.
\end{cases}
\]
Then
\[
\int_a^b |h(x)| \, dx \geq \int_a^b |h_1(x)| \, dx.
\]
Thus we may assume \( h = h_1 \), and the result reduces to a problem comparing areas of triangles. Note
\[
\int_a^{a_2} |h_1(x)| \, dx = \int_a^{a_1} g(x) \, dx = \frac{|f(a)|^2}{2N} = \frac{M}{N} \int_a^{a_2} |f(x)| \, dx.
\]
A similar inequality holds for the (possibly empty) interval \((a_3, b)\). Since \(|h_1| = |f|\) on \((a_2, a_3)\), the lower bound follows in this case with \( 1/C = M/N \).

The second case to consider is when \( \min g \geq 0 \). In this case
\[
\int_a^b |h(x)| \, dx \geq \int_a^b |g(x)|
\]
and the latter integral is at least one half of the area of the trapezoid with base \([a, b]\) and sides of height \( f(a) \) and \( f(b) \). The area of the trapezoid is larger than the area under the curve \( y = f(x) \) and hence the lower bound follows in this case with \( 1/C = 1/2 \).

Finally, if \( f(a) < 0 \) and \( f(b) < 0 \), then \(|h| \geq |f|\) and the lower bound holds with \( C = 1 \). \( \square \)

Applying Lemma 18 in the context of strip regions, we have
\[
\text{Area}(S \setminus \Omega_M) + \text{Area}(\Omega_M \setminus S) < +\infty
\]

23
holds for some $M$ if and only if it holds for all $M > 0$.

References.


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