Math 536 Homework #6

Spring 2013

1. For $z, w \in \mathbb{D}$, let

$$\rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|.$$

Remark: The problems below can be done by calculation, but some calculations become much shorter and easier with the aid of a good observation.

a. Show that if L is a Möbius map of $\mathbb{D}$ onto $\mathbb{D}$, then $\rho(L(z), L(w)) = \rho(z,w)$.

b. Prove the "world's greatest equality":

$$1-\rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2}.$$

c. Prove $\rho(|z|,|w|) \le \rho(z,w) \le \rho(|z|,-|w|)$.

d. Show $\rho(z,w)$ is a metric on $\mathbb{D}$.

e. Find the shortest path in this metric from 0 to r, where $r > 0$. Deduce a description of the shortest path from a to b, for $a, b \in \mathbb{D}$. These are called "geodesics". One can form a new metric, called the path metric, by taking the distance between two points to be that shortest path between the points. In the present case, the path metric is not equal to the original metric. The path metric is the so-called hyperbolic metric. It satisfies: $d = \log(1+\rho)/(1-\rho)$.

f. Show that $\log(1+\rho)/(1-\rho)$ is also a metric on $\mathbb{D}$. It is called the hyperbolic, or Poincaré, metric. The metric ρ is called the pseudohyperbolic metric on $\mathbb{D}$.

g. For $\varepsilon > 0$, prove that $B = \{z : \rho(z,a) < \varepsilon\}$ is a ball. Then find the Euclidean center and radius.

h. For $a, b \in \mathbb{D}$, $a \ne b$, let $C_{\{a,b\}} = \{z : \rho(z,a) = \rho(z,b)\}$. Without any explicit calculation show that $C_{\{a,b\}}$ must be a circular arc orthogonal to the unit circle. Hint: First consider the case $a = r > 0$, and $b = -r$.

i. Find the Euclidean center and radius of $C_a \equiv C_{\{0,a\}}$. Draw a picture.

j. When is $C_a \cap C_b = \emptyset$?

k. If L is a Möbius map of $\mathbb{D}$ onto $\mathbb{D}$, show $|L(0)| = |L^{-1}(0)|$. Conversely, given $a, b \in \mathbb{D}$ with $|a| = |b|$, show that there is a Möbius map L with $a = L(0)$ and $b = L^{-1}(0)$. Find L explicitly.

l. Suppose L is a Möbius map of $\mathbb{D}$ onto $\mathbb{D}$. Let R_1 denote reflection about the diameter $C_{\{a,b\}}$, where $a = L(0)$ and $b = L^{-1}(0)$. Let R_2 denote reflection about $C_{\{0,L(0)\}}$. Show that $L = R_2 \circ R_1$.

2. Suppose $a_1, \ldots, a_n \in \mathbb{D}$ and suppose $\text{Im} a_j > 0$. Let $C_k = C_{\{0,a_k\}}$ and suppose $D_k \cap D_j = \emptyset$ where D_j is the open disk bounded by C_j, and suppose $C_j \cap [-1,1] = \emptyset$ for $j \neq k$. Let $a_{-j} = \overline{a_j}$ and let L_j be the Möbius map with $L_j(0) = a_j$ and $L_j^{-1}(0) = a_{-j}$. Let $\mathcal{F}$ denote the subset of $\mathbb{D}$ given by

$$\mathcal{F} = \{z \in \mathbb{D} : \rho(z,0) < \rho(z,a_j), \text{ for all } j\}.$$

a. Prove that $\overline{\mathcal{F}}$ can be made into a Riemann surface W by identifying C_j and C_{-j}. The set $\mathcal{F}$ is called a normal fundamental domain.

b. Show that $\mathbb{D}$ is the universal covering surface of W by repeatedly reflecting $\mathcal{F}$ about the arcs C_j and their images.

c. Show that $L_j, j = 1, \ldots, n$ are generators of the Fuchsian group $\mathcal{G}$ for W.

d. Show that $\mathcal{G}$ is a free group on n generators. The reflections give one way to "see" a free group on n generators.

e. Let φ be a conformal map of $\mathcal{F} \cap \{z : \text{Im} z > 0\}$ onto the upper half plane with $\varphi(0) = \infty$. Let $E = \mathbb{R} \setminus \cup I_j$, where $I_j = \varphi(C_{\{0,a_j\}})$. Prove that W is conformally equivalent to $\mathbb{C}^* \setminus E$.