Math 536 Homework #5

Spring 2013

1. Assume that $\partial\Omega$ consists of finitely many analytic curves. Let $g = g(z, w)$ be Green's function for Ω with pole at w and suppose f is integrable on $\partial\Omega$ and continuous at $\zeta_0 \in \partial\Omega$. Prove

$$u(w) = \frac{1}{2\pi}\int_{\partial\Omega} f(\zeta)\frac{\partial g}{\partial\eta}(\zeta)|d\zeta|$$

is harmonic on Ω and

$$\lim_{w\to\zeta_0} u(w) = f(\zeta_0).$$

Hint: See Exercise VIII.3a.

2. Suppose $p(z) = a_k z^k + \ldots + a_0$ is a polynomial. Set $\Omega = \{|p| > \varepsilon\}$. Near ∞ we can write Green's function with pole at ∞ as $g(z, \infty) = \log|z| + \gamma(\partial\Omega) + o(1)$. The constant $\gamma(\partial\Omega)$ is called Robin's constant for $\partial\Omega$. Prove:

(a) $g_\Omega(z, \infty) = \frac{1}{k}\log\left|\frac{p(z)}{\varepsilon}\right|$.

(b) $\gamma(\partial\Omega) = \frac{1}{k}\log\left|\frac{a_k}{\varepsilon}\right|$.

(c) $-\frac{\partial g}{\partial \mathrm{n}}\frac{ds}{2\pi} = \frac{1}{k}\frac{p'(z)}{p(z)}\frac{dz}{2\pi i}$.

(d) The harmonic measure of a component J of $\partial\Omega$ is defined to be the solution of the Dirichlet problem with boundary data equal to 1 on J and equal to 0 on $\partial\Omega \setminus J$. Prove that the harmonic measure at ∞ for J is equal to the number of zeros of p in the bounded component of $C \setminus J$, divided by k.

(e) Formulate and prove similar results for rational functions.

3. Let Ω be a finitely connected Jordan domain such that $\infty \in \Omega$ and every component of $\partial\Omega$ is a C^2 curve. Let $G(z) = g_\Omega(z, \infty)$ and let $\{z_j\}$ be the (finite) set of critical points of G in Ω. Prove

$$\frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial G}{\partial \mathrm{n}}\log\left|\frac{\partial G}{\partial \mathrm{n}}\right|ds = \gamma(\partial\Omega) + \sum G(z_j)$$

where $\gamma(\partial\Omega)$ is Robin's constant for $\partial\Omega$. Hint: Use Green's theorem and the Taylor expansion of $G_x - iG_y$ near each critical point z_j.

The integral above is called the "entropy" of harmonic measure.

4. a. Suppose Ω is simply connected and bounded by an analytic curve. It is known that if J is an open subarc of $\partial\Omega$ and if h is bounded on $\partial\Omega$ with $h \in C^2(J)$ then the solution u to the

Dirichlet problem with boundary data h satisfies $u \in C^1(\Omega \cup J)$ (for this problem, you may assume this fact). Suppose $f \in C^1(\partial\Omega)$, with $\int_{\partial\Omega} f(\zeta)|d\zeta| = 0$. Prove that there is a function u harmonic on Ω, $u \in C^1(\overline{\Omega})$ with $\frac{\partial u}{\partial \eta} = f$ on $\partial\Omega$. Hint: Use the Cauchy-Riemann equations. See also Exercise IX.8.

b. Find the Neumann kernel $N(z, \zeta)$ where

$$u(z) = \int_{\partial\Omega} N(z, \zeta) f(\zeta)|d\zeta|,$$

in terms of Green's function, where f and u are from part a.

c. Find the Neumann kernel explicitly on the unit disk.