Math 536 Homework #4

Spring 2012

The first three problems are "A" exercises. You are to do them, but not hand them in.

1. Prove from scratch that the equivalence classes of curves beginning and ending at z_0 and contained in a region Ω form a group. i.e. verify that homotopy is an equivalence relation, that there is a multiplication defined on the homotopy classes which gives the group multiplication. Identify the unit, the inverse of an element of the group, associativity of multiplication, etc. Show that the group is independent of the choice of base point z_0. Finally show that this group is a topological invariant: If F is a homeomorphism of Ω onto Ω_1 then the fundamental groups for Ω and Ω_1 are isomorphic. Look at the proofs and decide what assumptions you need on Ω, i.e. if Ω is an abstract set, what assumptions are needed to make all of these statements still hold?

2. (a) Find the fundamental group of $\mathbb{D} \setminus \{0\}$ (prove it). (b) Show that a doubly connected region has the same fundamental group as $\mathbb{D} \setminus \{0\}$.

3. Prove that if f_a is a germ at a and if f_a has an analytic continuation along a curve γ, with $\gamma(0) = a$, then the continuation along γ is unique.

The following are "B" exercises, to be handed in next Wednesday.

1. (The modular function). Let S be the region $\{z = x + iy : |x| < 1, \quad y > 0, \text{ and } x^2 + y^2 > 1\}$. Let π be the conformal map of S onto the upper half plane H with the property that $\pi(-1) = 0$, $\pi(1) = 1$ and so that $|\pi(z)| \to \infty$ as $z \in S \to \infty$. Reflect π across each boundary curve repeatedly. Show that these reflections fill the upper half plane. Show that any curve in $\mathbb{C} \setminus \{0, 1\}$ which begins at $\frac{1}{2}$ lifts to a unique curve in $\mathbb{H}$ which begins at $i = \pi^{-1}(\frac{1}{2}) \in \partial S$. Prove that the lifted curve is closed if and only if the original curve is homotopic to a point. This is an explicit construction of the simply connected covering surface of $\mathbb{C} \setminus \{0, 1\}$.

2. Suppose g is entire and g omits the values 0 and 1. Show that $\pi^{-1} \circ g$ can be defined to be entire, where π is defined in problem 1. Conclude that g is constant. (This was the original proof of Picard's (little) theorem.)

3. Let $\pi(z) = e^{\frac{z+1}{z-1}}$, for $z \in \mathbb{D}$. Show that π maps $\mathbb{D}$ onto $\mathbb{D} \setminus \{0\}$. Show that if $z_0 \in \mathbb{D} \setminus \{0\}$, then there is a small ball B containing z_0 such that $\pi^{-1}(B)$ consists of countably many regions on each

of which π is a homeomorphism. This is an explicit construction of the covering map for $\mathbb{D} \setminus \{0\}$ guaranteed by the theorem that will be proved in class.

4. (a) Let π be the map given in problem 3. Find an explicit conformal map L of $\mathbb{D}$ onto $\mathbb{D}$ such that $\pi(z) = \pi(w)$ if and only if $z = L^{(n)}(w)$ for some integer n, where $L^{(n)}$ is the n-fold composition of L if n is positive and the $-n$-fold composition of the inverse of L if n is negative.

(b) Let $Z = \pi^{-1}(\pi(0))$. Let $\mathcal{F} = \{z \in \mathbb{D} : \rho(z, 0) < \rho(z, a) \text{ for all } a \in Z \setminus \{0\}\}$, where ρ is the pseudohyperbolic metric given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|.$$

Describe $\mathcal{F}$ geometrically.

(c) Show that π is a one-to-one conformal map of $\mathcal{F}$ onto the disk with a slit from 0 to $\partial\mathbb{D}$ removed. Identifying two edges of the boundary of $\mathcal{F}$ we obtain a "conformal" copy of $\mathbb{D} \setminus \{0\}$. The group $\mathcal{G} = \{L^{(n)}\}$ is called the Fuchsian group for $\mathbb{D} \setminus \{0\}$ and the region $\mathcal{F}$ is called the normal fundamental domain.

(d) Prove that f is analytic on $\mathbb{D} \setminus \{0\}$ if and only if there is an analytic funtion g defined on $\mathbb{D}$ with $g \circ L = g$ on $\mathbb{D}$ and $g = f \circ \pi$. A similar statement holds for meromorphic, harmonic and subharmonic functions. This is a way to transfer function theory on the domain $\mathbb{D} \setminus \{0\}$ to $\mathbb{D}$.

5. Repeat problem 4 using the map π constructed in problem 1, by composing with a map from the disk to the upper half plane sending 0 to i. In this case there will be two maps L_1 and L_2 of the disk onto the disk. They generate the Fuchsian group. In fancier words, $\mathbb{C} \setminus \{0, 1\}$ is conformally equivalent to the disk modulo the Fuchsian group. The normal fundamental domain consists of two copies of S. Part of this problem is to figure out the right statements to prove. Hint: reflections are conjugate analytic.