**Math 536 Homework #3**

Spring 2012

1. This problem gives another way of finding the harmonic conjugate of a harmonic function. Suppose $f(z) = \sum_n a_n(z-z_0)^n$ is analytic in a neighborhood of $z_0$. Set $u(x,y) = Ref(z)$ where $z = x+iy$. Show that

$$f(z) = 2u\Big(\frac{z-z_0}{2}+x_0, \frac{z-z_0}{2i}+y_0\Big) - u(x_0,y_0) + iC$$

where $z_0 = x_0 + iy_0$, and $C$ is a real constant. Hint: WLOG $z_0 = 0, f(z_0) = 0$. Consider analytic functions of two complex variables (locally convergent power series) and use the convergence of $\sum |a_n|(|x|+|y|)^n$ for $|x|,|y| < \delta$. Apply this idea to $u(x,y) = e^{-nx}\sin ny$ to find an analytic function with real part equal to $u$. Another approach is to work directly with the Poisson kernel $(1-|z|^2)/|e^{it}-z|^2$. (rewrite the latter in terms of $x$ and $y$ without absolute values.)

2. a. Prove a "Schwarz's Lemma" for harmonic functions: If $u$ is real-valued and harmonic in $\mathbb{D}$ with $|u| \le 1$ and $u(0) = 0$, then

$$|u(z)| \le \frac{4}{\pi}\arctan|z|.$$

b. Given $z_1, z_2 \in \mathbb{D}$ and $v_1, v_2 \in \mathbb{R}$. Find a necessary and sufficient condition for the existence of a harmonic function $u$ defined on $\mathbb{D}$ and bounded by 1 such that $u(z_j) = v_j$, $j = 1, 2$.

If "harmonic" is relaced by "analytic" in part b, then the result is sometimes called the invariant form of Schwarz's lemma. The solution for $n$ points and values for analytic functions is known as Pick's theorem. It is an open problem to do this for 3 points and values for harmonic functions.

3. Consider

$$f(z) = \int_0^z e^{w^2}dw.$$

Prove $f$ is entire and maps the plane onto the plane. (Hint: $f'$ is even)

Show that $f$ is locally one-to-one but not one-to-one in the plane.

3. (a) State the Monodromy Theorem.

(b) Suppose that $p$ is a polynomial and suppose $\Omega$ is a simply connected domain with the property that if $p'(z) = 0$ then $p(z) \notin \Omega$. Prove that there is an analytic function $g$ defined on $\Omega$ so that $p(g(z)) = z$.

(c) Suppose $f$ is an entire function which maps the plane onto the plane and for which $f'$ is never equal to 0. The example in problem 3 shows that $f$ does not necessarily have an inverse on $\mathbb{C}$. Explain which statement(s) is (are) wrong in the following argument:

> If $\gamma$ is any curve in the plane, then a local inverse of $f$ can be defined in a neighborhood of each point of $\gamma$. If $\gamma$ is a curve from 0 to $z$ then $\gamma$ is compact and can be covered by finitely many discs, such that $f$ has an inverse on each. This provides an analytic continuation of $f^{-1}$ from 0 to $z$ along $\gamma$. Since the plane is simply connected, a global inverse $f^{-1}$ of $f$ can be defined in the plane by the monodromy theorem.

5. Suppose $f$ and $g$ are entire and $f^2 + g^2 = 1$. Prove there exists an entire function $h$ so that $f(z) = \cos h(z)$ and $g(z) = \sin h(z)$.

6. Construct a curve in $\Omega = \mathbb{C} \setminus \{0, 1\}$ which is homologous to 0 (i.e. the winding number of the curve about each point in the complement of $\Omega$ is zero) but is not homotopic to a point. You may use the example given in class, but you must prove these statements. Hint: consider the conformal map of an equilateral triangle onto the upper half plane chosen so that the vertices are mapped to $0, 1, \infty$. Reflect as much as possible.

7. Let $Log$ denote the principal branch of the logarithm, and let $t \in (0, 1)$. The discontinuities of

$$Log(z) - Log(z - t)$$

cancel on $(-\infty, 0)$, so the above function has an analytic continuation $F_t(z)$ to $\mathbb{C} \setminus [0, t]$. Let $\Gamma$ be the unit circle with positive orientation. Compute

$$\int_\Gamma F_t(z) dz.$$

8. Suppose $f$ and $g$ are nonconstant entire functions and suppose $f \circ g$ is a polynomial. Show that both $f$ and $g$ are polynomials.

9. Prove that Harnack's princple (about an increasing sequence of harmonic functions) holds on any Riemann surface.