Math 536 Homework #2

Spring 2012

1. a. Suppose u is bounded and harmonic in a bounded region Ω. Suppose

$$\limsup_{z\to\zeta} u(z) \le M$$

for all $\zeta \in \partial\Omega \setminus F$, where $F = \{\zeta_1, \ldots, \zeta_n\}$ is a finite subset of $\partial\Omega$. Prove $u \le M$. Hint: consider $u + \varepsilon \sum \log|z-\zeta_j|/d$ where d is the diameter of Ω.

b. Prove the same result assuming only that u is bounded and harmonic and F is a finite subset of $\partial\Omega$, but $F \ne \partial\Omega$. Hint: delete a small neighborhood of a point in $\partial\Omega \setminus F$ and apply part a.

c. Give an example of an unbounded u where the conclusion to part a is false.

This exercise is called Lindelöf's maximum principle.

2. Suppose that $u(e^{i\theta}) \in L^1$ has a jump discontinuity at $\theta = \theta_0$. Examine the behavior of the Poisson integral $Pu(z)$ as $z \to e^{i\theta}$. Hint: $\arg(1-z)$ has a jump discontinuity at 1.

3. Suppose that u is harmonic in $\{z : 0 < |z| < 1\}$ and

$$\int_0^{2\pi} u\left(\frac{e^{it}}{2}\right)\frac{dt}{2\pi} = 1 \text{ and } \int_0^{2\pi} u\left(\frac{3e^{it}}{4}\right)\frac{dt}{2\pi} = 2.$$

Find $M(r) = \int_0^{2\pi} u(re^{it})\frac{dt}{2\pi}$ explicitly. The notation means that M is a function of the polar coordiates r and t which happens to be independent of t. Hint: Prove M is harmonic.

4. Suppose u is bounded and harmonic in $\{z : 0 < |z| < 2\}$. Prove that u has a harmonic extension to $\{z : |z| < 2\}$.

5. Find a harmonic function u in $\mathbf{D} = \{z : |z| < 1\}$ such that $0 < u < 1$ and so that

$$\lim_{z\to\zeta\in\partial\mathbf{D}} u(z) = \begin{cases} 1 & \text{if } Im\ \zeta > 0 \\ 0 & \text{if } Im\ \zeta < 0 \end{cases}.$$

This function u is called the "harmonic measure" for the upper half-circle. Write out u explicitly-not just an integral. It is useful for estimating harmonic or subharmonic functions: Example: Suppose f is analytic in $\mathbf{D}$ and

$$\limsup_{z\to\zeta\in\partial\mathbf{D}} |f(z)| \le \begin{cases} M & \text{if } Im\ \zeta \ge 0 \\ 1 & \text{if } Im\ \zeta < 0 \end{cases}.$$

then

$$|f(z)| \leq e^{Mu(z)}$$

for $z \in \mathbf{D}$. (cf. Hadamard's three circles theorem)

6. **The boundary Harnack inequality**. Let u and v be positive harmonic functions on $\mathbb{D}$ with $u(0) = v(0)$, let $I \subset \partial\mathbb{D}$ be an open arc and assume

$$\lim_{z \to \zeta} u(z) = \lim_{z \to \zeta} v(z) = 0$$

for all $\zeta \in I$. Prove that for every compact $K \subset \mathbb{D} \cup I$ there is a constant $C(K)$ independent of u and v such that on $K \cap \mathbb{D}$

$$\frac{1}{C(K)} \leq \frac{u(z)}{v(z)} \leq C(K).$$

Hint: Using a conformal map of a subregion of $\mathbb{D}$ onto $\mathbb{D}$, you may assume u and v are continuous on $\overline{\mathbb{D}}$. Estimate the Poisson kernel on $\partial\mathbb{D} \setminus I$.