§3. Carathéodory's Theorem

Let Ω be a simply connected domain in the extended plane $\mathbb{C}^*$. We say Ω is a **Jordan domain** if $\Gamma = \partial\Omega$ is a Jordan curve in $\mathbb{C}^*$.

Theorem 3.1. (Carathéodory). *Let φ be a conformal mapping from the unit disc $\mathbb{D}$ onto a Jordan domain Ω. Then φ has a continuous extension to $\overline{\mathbb{D}}$, and the extension is a one-to-one map from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$.*

Because φ maps $\mathbb{D}$ onto Ω, the continuous extension (also denoted by φ) must map $\partial\mathbb{D}$ onto $\Gamma = \partial\Omega$, and because φ is one-to-one on $\partial\mathbb{D}$, $\varphi(e^{i\theta})$ parameterizes the Jordan curve Γ.

Before we prove Carathéodory's theorem, we use it to solve the Dirichlet problem on a Jordan domain Ω. Let f be Borel function on Γ such that $f \circ \varphi$ is integrable on $\partial\mathbb{D}$. If $w = \varphi^{-1}(z)$, then

$$u(z) \equiv u_f(z) = \int_0^{2\pi} f \circ \varphi(e^{i\theta}) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi} \tag{3.1}$$

is harmonic on Ω, and by Theorem 3.1 and Theorem 1.3,

$$\lim_{\Omega \ni z \to \zeta} u(z) = f(\zeta) \tag{3.2}$$

whenever $\varphi^{-1}(\zeta) \in \partial\mathbb{D}$ is a point of continuity of $f \circ \varphi$. In particular, if f is continuous on Γ then (3.2) holds for all $\zeta \in \Gamma$ and $u(z) = u_f(z)$ solves the **Dirichlet problem** for f on Ω.

If f is a bounded Borel function on Γ, then $f \circ \varphi$ is Borel, and the integral (3.1) is defined.

Proof of Theorem 3.1. We may assume Ω is bounded. Fix $\zeta \in \partial\mathbb{D}$. First we show φ has a continuous extension at ζ. Let $0 < \delta < 1$ and write

$$\gamma_\delta = \mathbb{D} \cap \{z : |z - \zeta| = \delta\}.$$

Then $\varphi(\gamma_\delta)$ is a Jordan arc having length

$$L(\delta) = \int_{\gamma_\delta} |\varphi'(z)| ds.$$

By the Cauchy-Schwarz inequality

$$L^2(\delta) \leq \pi\delta \int_{\gamma_\delta} |\varphi'(z)|^2 ds,$$

so that for $\rho < 1$,

$$\int_0^\rho \frac{L^2(\delta)}{\delta} d\delta \leq \pi \int\int_{\mathbb{D}\cap B(\zeta,\rho)} |\varphi'(z)|^2 dx\, dy$$

$$= \pi \text{Area}\,(\varphi(\mathbb{D} \cap B(\zeta, \rho)) < \infty. \tag{3.5}$$

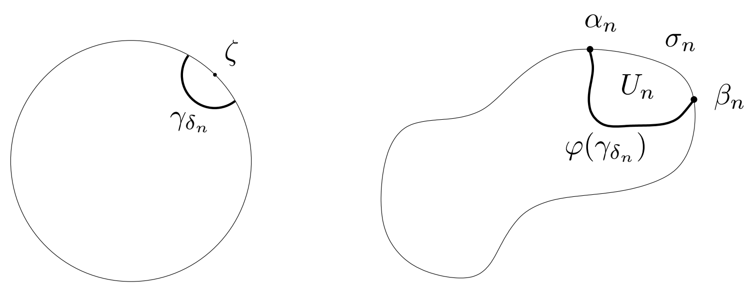

Figure I.5 Crosscuts γ_{δ_n} and $\varphi(\gamma_{\delta_n})$.

Therefore there is a sequence $\delta_n \downarrow 0$ such that $L(\delta_n) \to 0$. When $L(\delta_n) < \infty$, the curve $\varphi(\gamma_{\delta_n})$ has endpoints α_n, β_n, and both of these endpoints must lie on $\Gamma = \partial\Omega$. Indeed, if $\alpha_n \in \Omega$, then some point near α_n has two distinct preimages in $\mathbb{D}$ because φ maps $\mathbb{D}$ onto Ω, and that is impossible because φ is one-to-one. Furthermore,

$$|\alpha_n - \beta_n| \leq L(\delta_n) \to 0. \tag{3.6}$$

Let σ_n be that closed subarc of Γ having endpoints α_n and β_n and having smaller diameter. Then (3.6) implies

$$\text{diam}(\sigma_n) \to 0,$$

because Γ is homeomorphic to the circle. By the Jordan curve theorem the curve $\sigma_n \cup \varphi(\gamma_{\delta_n})$ divides the plane into two (connected, open) regions, and one of these regions, say U_n, is bounded. Then $U_n \subset \Omega$, because $\mathbb{C}^* \setminus \overline{\Omega}$ is arcwise connected. Since

$$\text{diam}(\partial U_n) = \text{diam}\big(\sigma_n \cup \varphi(\gamma_{\delta_n})\big) \to 0,$$

we conclude that

$$\operatorname{diam}(U_n) \to 0. \tag{3.7}$$

Set $D_n = \mathbb{D} \cap \{z : |z - \zeta| < \delta_n\}$. We claim that for large n, $\varphi(D_n) = U_n$. If not, then by connectedness $\varphi(\mathbb{D} \setminus \overline{D}_n) = U_n$ and

$$\operatorname{diam}(U_n) \geq \operatorname{diam}(\varphi(B(0, 1/2)) > 0,$$

which contradicts (3.7). Therefore $\operatorname{diam}(\varphi(D_n)) \to 0$ and $\bigcap \overline{\varphi(D_n)}$ consists of a single point, because $\varphi(D_{n+1}) \subset \varphi(D_n)$. That means φ has a continuous extension to $\{\zeta\} \cup \mathbb{D}$. It is an exercise to show that the union over ζ of these extensions defines a continuous map on $\overline{\mathbb{D}}$.

Let φ also denote the extension $\varphi : \overline{\mathbb{D}} \to \overline{\Omega}$. Since $\varphi(\mathbb{D}) = \Omega$, φ maps $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. To show φ is one-to-one, suppose $\varphi(\zeta_1) = \varphi(\zeta_2)$ but $\zeta_1 \neq \zeta_2$. The argument used to show $\alpha_n \in \Gamma$ also shows that $\varphi(\partial\mathbb{D}) = \Gamma$, and so we can assume $\zeta_j \in \partial\mathbb{D}$, $j = 1, 2$. The Jordan curve

$$\big\{\varphi(r\zeta_1) : 0 \leq r \leq 1\big\} \ \cup \ \big\{\varphi(r\zeta_2) : 0 \leq r \leq 1\big\}$$

bounds a bounded domain $W \subset \Omega$, and then $\varphi^{-1}(W)$ is one of the two components of

$$\mathbb{D} \setminus \Big(\{r\zeta_1 : 0 \leq r \leq 1\} \cup \{r\zeta_2 : 0 \leq r \leq 1\}\Big).$$

But since $\varphi(\partial\mathbb{D}) \subset \Gamma$,

$$\varphi(\partial\mathbb{D} \cap \partial\varphi^{-1}(W)) \subset \partial W \cap \partial\Omega = \{\varphi(\zeta_1)\}$$

and φ is constant on an arc of $\partial\mathbb{D}$. It follows that φ is constant, either by Schwarz reflection principle or by the Jensen formula, and this contradiction shows $\varphi(\zeta_1) \neq \varphi(\zeta_2)$. $\square$

One can also prove φ is one-to-one by repeating for φ^{-1} the proof that φ is continuous.

The Cauchy-Schwarz trick used to prove (3.5) is known as a *length-area* argument. The length-area method is the cornerstone of the theory of extremal length.

Corollary. *If $h : \partial\mathbb{D} \to \mathbb{C}$ is a homeomorphism then h extends to be a homeomorphism of $\mathbb{C}$ onto $\mathbb{C}$.*

Proof. Suppose f and g are Riemann maps of the interior and exterior of the disk onto the inner and outer domains of J, with $g(\infty) = \infty$. For $|\zeta| = 1$, set

$$F(r\zeta) = \begin{cases} f(rf^{-1}(h(\zeta))) & \text{for } r \leq 1 \\ g(rg^{-1}(h(\zeta))) & \text{for } r \geq 1. \end{cases}$$

□

This Corollary fails in higher dimensions. Alexander's Horned Sphere is a homeomorphic image of the ball in $\mathbb{R}^3$. The boundary is the homeomorphic image of the sphere, but the unbounded component of the complement is not simply connected in the sense that there are curves in the complement that are not homotopic in the complement to a point. In particular the homeomorphism cannot be extended to a homeomorphism of $\mathbb{R}^3$.

Exercises:

1. A compact set K is **locally connected** if whenever U is a relatively open subset of K and $z \in U \subset K$ there is a relatively open subset V of K such that V is connected and $z \in V \subset U$. Let Ω be a simply connected domain such that $\partial\Omega$ contains at least two points. Prove $\partial\Omega$ is locally connected if and only if the Riemann map $\varphi : \mathbb{D} \to \Omega$ extends continuously to $\overline{\mathbb{D}}$. Hint: For one direction, use the uniform continuity of φ. For the other direction follow the proof of Carathéodory's theorem.

2. (a) Let Ω be a simply connected domain and let $\sigma \subset \Omega$ be a **crosscut**, that is, a Jordan arc in Ω having distinct endpoints in $\partial\Omega$. Prove that $\Omega \setminus \sigma$ has two components Ω_1 and Ω_2, each simply connected, and $\beta_j = \partial\Omega_j \setminus \sigma$ is connected.

(b) Let Ω be simply connected, let $\psi : \Omega \to \mathbb{D}$ be conformal and fix $z \in \Omega$ and $\zeta \in \partial\Omega$. Assume ζ and z can be separated by a sequence of crosscuts $\gamma_n \subset \Omega$ such that $\text{length}(\gamma_n) \to 0$. Let U_n be the component of $\Omega \setminus \gamma_n$ such that $z \notin U_n$. Prove that $\psi(U_n) \to \alpha \in \partial\mathbb{D}$.

3. Find and read a construction of Alexander's Horned Sphere and the proof of the claim after the Corollary above. Give a reference to the proof, and say whether or not you were able to follow the proof. Note also that the notion of simply connected in $\mathbb{R}^3$ is not the same as having a connected complement.