Complex Analysis Preliminary Examination

September, 2004

Directions: There are eight problems. Four correct solutions will be a clear pass. Only four problems will be read on any paper; indicate which four you want to have considered.

The following standard notations are used:
$\mathbb{R}$ = the field of real numbers with its usual topology
$\mathbb{C}$ = the field of complex numbers with its usual topology.
$\mathbb{U}$=the open unit disc in $\mathbb{C}$.
$\overline{\mathbb{U}}$ = the closed unit disc in $\mathbb{C}$.

Also, the expressions *holomorphic function, analytic functions*, and *complex-analytic function* are used synonymously here.

****

1. Let f be a function holomorphic on a neighborhood of $\overline{\mathbb{U}}$.

a) Evaluate the integrals

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - z}\,d\zeta \text{ and } \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - \frac{1}{\bar{z}}}\,d\zeta$$

for$|z| < 1$.

b) Use (a) or some other method to prove that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}f(e^{i\phi})\,d\phi$$

for $0 \le r < 1$.

2. a) State the Weierstrass factorization theorem on the existence of entire functions with prescribed zeros with prescribed orders. Then state the Mittag-Leffler theorem on the existence of meromorphic functions on the plane with prescribed poles and prescribed principal parts.

b) From the theorems stated in a) deduce the following theorem: If $\{z_n\}_{n=1}^{\infty}$ is a sequence of distinct points in the plane with no finite limit point, and if $\{c_n\}_{n=1}^{\infty}$ is a sequence of complex numbers, then there is an entire function f with $f(z_n) = c_n$ for all n.

3. Show that for $c > 0$

$$\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z}\,dz = \begin{cases} 2\pi i & \text{if } a > 1 \\ 0 & \text{if } 0 < a < 1. \end{cases}$$

The path of integration is the vertical line $Re\,z = c$ traversed in the upward direction.

4. Let f be holomorphic in $\mathbb{U}$.

a) Show that the quantity

$$\int_{|z|=r} |f(z)|^2 \frac{dz}{iz}$$

is a convex function of r on the interval $(0,1)$. You may use the fact that a function h on (a,b) is convex if it has two continuous derivatives and $h'' \geq 0$ on (a,b).

b) Prove the same result for the integrals

$$\int_{|z|=r} |f(z)|^k \frac{dz}{iz},$$

for k an even positive integer.

5. Let P be a polynomial with complex coefficients, P not identically zero.

a) Prove that the series

$$\sum_{n=0}^{\infty} P(n) z^n$$

converges in $\mathbb{U}$ and in no larger open set.

b) Show that if f is the sum of this series, then f continues into the whole Riemann sphere as a meromorphic function.

6. Show that there is an entire function f with the property that for every entire function g, for every $\varepsilon > 0$, and for every compact set K in the plane, there is a $c > 0$ such that

$$\sup_{z \in K} |g(z) - f(z+c)| < \varepsilon$$

7. Give explicitly a function that maps the set G described below 1-1 and conformally onto the unit disc. (It will suffice to give explicitly finitely many conformal mappings whose composition effects the desired mapping. You must, of course, justify each step in your process.)

$$G = \{z \in \mathbb{C} : z = x + iy,\ |z| < 1 \text{ and either } x > 0 \text{ or } y^2 > 3x^2\}$$

8. Let w_1 and w_2 be complex numbers that are linearly independent over $\mathbb{R}$ so that neither is a nonzero real multiple of the other.

a) Prove that the only entire functions f that satisfy the functional equations $f(z+w_1) = f(z)$ and $f(z+w_2) = f(z)$ for all $z \in \mathbb{C}$ are the constants.

b) Set $w_{m,n} = m + in$ for all integers m and n. Define f by

$$f(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \right\}.$$

Show that f is a meromorphic function on $\mathbb{C}$ that satisfies the equations $f(z+i) = f(z) = f(z+1)$ and has a pole of order two at the origin.