1. Let $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

(a) Show: $\Psi'(z) + \Psi'(z + \frac{1}{2}) = 4\Psi(2z)$.

(b) Integrate (a) twice to obtain

$$\Gamma(2z) = e^{az+b}(z)(z + \frac{1}{2})$$

for some constants $a$ and $b$. (Careful: $\log \Gamma(z)$ is only defined locally.)

(c) Find $a$ and $b$. The resulting formula is called **Legendre’s duplication formula**.

2. Stirling’s formula for the gamma function...used in combinatorics, physics,

(a) Suppose $x > \frac{1}{2}$. Substitute $t = (\sqrt{x} + v)^2$ in the integral formula (4.6) for $\Gamma(x)$ to show that

$$\frac{\Gamma(x)e^{x\sqrt{x}}}{x^x} = 2 \int_{-\infty}^{\infty} \varphi_x(v)e^{-v^2} dv,$$

where

$$\varphi_x(v) = \begin{cases} 
0 & \text{if } v \leq -\sqrt{x} \\
e^{-2vx^2} \left(1 + \frac{v}{\sqrt{x}}\right)^{2x-1} & \text{if } v \geq -\sqrt{x}.
\end{cases}$$

(b) Use the Dominated convergence theorem to show that

$$\lim_{x \to \infty} \frac{\Gamma(x)e^{x\sqrt{x}}}{x^x} = 2 \int_{-\infty}^{\infty} e^{-2v^2} dv = \sqrt{2\pi}.$$ 

Hint: find an upper bound for $\log \varphi_x(v)$ as a function of $v$.

(c) Use part (b) to estimate $n!$.

3. Find all non-zero residues of the following functions:

(a) $\frac{z-1}{(z+1)^2(z-2)}$

(b) $\frac{z^2-2z}{(z+1)^2(z^2+4)}$

(c) $e^z \csc^2 z$

(d) $ze^{-1/z^2}$

(e) $\cot \pi z$

(f) $\frac{1}{z^6}$

4. Let $C$ be the circle of radius 3 centered at 0, oriented in the positive sense. Find

$$\int_C \frac{e^{\lambda z}}{(z+4)(z-1)^2(z^2+4z+5)} dz.$$