1. Suppose $f$ is analytic in a neighborhood of $D$ and that $|f(z)| < 1$ on $\partial D$. Using Rouché’s theorem show that there exists one and only one point $z_0 \in D$ such that $f(z_0) = z_0$.

2. Let $f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots$ be analytic in $D$. Suppose 
   
   \[ c = |a_2| + |a_3| + |a_4| + \ldots < 1. \]

   Show that if $|w| < 1 - c$ then there is exactly one $z \in D$ with $f(z) = w$.

3. How many zeros does $p(z) = 3z^5 + 21z^4 + 5z^3 + 6z + 7$ have in $D$? How many zeros in $\{z : 1 < |z| < 2\}$? Choose coefficients of a fourth degree polynomial randomly and find out how many zeros in $D$, using the algorithm in Appendix A.

4. Prove that the number of roots of the equation 
   
   \[ z^{2n} + \alpha z^{2n-1} + 2 = 0, \]

   where $\alpha, \beta$ are real and nonzero, and $n$ is a natural number, that have positive real part is equal to $n$ if $n$ is even. If $n$ is odd, their number is $n - 1$ for $\alpha > 0$ and $n + 1$ for $\alpha < 0$.

   Hint: See what happens to 
   
   \[ z^{2n} + \alpha z^{2n-1} + \beta^2 \]

   as $z$ traces the boundary of a large half-disk.

5. Let $n$ be a positive integer and $a > 0$. Show that there exists $f$ analytic in a neighborhood of $z = 1$ such that $f(1) = 1$ and 

   \[ af(z)^{n+1} + (1 - a)f(z)^n = z \]

   in a neighborhood of $z = 1$.

6. (this problem is complementary to problem 7 of HW #1). Suppose $f$ is analytic on an open set $U$ and $z_0 \in U$. Prove that there are points $z_1, z_2 \in U$ such that 

   \[ f'(z_0) = \frac{f(z_1) - f(z_2)}{z_1 - z_2}. \]

   Hint: consider $g(z) = z f'(z_0) - f(z)$. 
7. Prove that all of the zeros of the polynomial

\[ p(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_1 z + c_0 \]

lie in the disc centered at 0 with radius

\[ R = \sqrt{1 + |c_{n-1}|^2 + \ldots + |c_1|^2 + |c_0|^2}. \]

8. Let \( f(z) \) be an entire function with only finitely many zeroes. Define

\[ m(r) = \min_{|z|=r} |f(z)|. \]

Show that if \( f \) is not a polynomial then \( m(r) \to 0 \) as \( r \to \infty \).

9. Let \( P \) be a polynomial with complex coefficients, not identically zero. Prove that the series

\[ \sum_{n=0}^{\infty} P(n)z^n \]

converges in \( \mathbb{D} \) and in no larger open set. Show that if \( f \) is the sum of this series, then \( f \) can be extended to be meromorphic in \( \mathbb{C} \) such that the singularity at \( \infty \) is not essential. Find the function \( f \).

10. Suppose that \( \Omega \) is a bounded region in \( \mathbb{C} \) such that \( \partial \Omega \) is a finite union of disjoint (piecewise continuously differentiable) closed curves \( \Gamma_j, j = 1, \ldots, n \). Suppose that \( f \) is analytic on \( \overline{\Omega} \). Prove that \( f = \sum f_j \) where \( f_j \) is analytic on the component of \( \mathbb{C} \setminus \Gamma_j \) which contains \( \Omega \).