Math 534 Homework #3
Autumn 2009

1. Suppose \( f \) is analytic in \( \mathbb{C} \) and \( |f(z)| \leq M|z|^\alpha \) for \( |z| > R \), where \( \alpha \) is a non-negative real number. Prove \( f \) is a polynomial of degree \( \leq \alpha \).

2. Prove that if \( f \) is non-constant and analytic on all of \( \mathbb{C} \) then \( f(\mathbb{C}) \) is dense in \( \mathbb{C} \).

3. Let \( f \) be analytic in \( \mathbb{D} \) and satisfy \( |f(z)| \to 1 \) as \( |z| \to 1 \). Prove \( f \) is rational.

4. Suppose \( f \) and \( g \) are analytic in \( \mathbb{C} \) and \( |f(z)| \leq |g(z)| \) for all \( z \). Prove there exist a constant \( c \) so that \( f(z) = cg(z) \) for all \( z \).

5. Suppose \( f \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq M \) on \( \mathbb{D} \). Prove that the number of zeros of \( f \) in the disc of radius \( 1/4 \), centered at \( 0 \), does not exceed

\[
\frac{1}{\log 4} \log \left| \frac{M}{f(0)} \right|.
\]

6. Suppose \( f \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \) and \( f(0) = 1/2 \). Prove that \( |f(1/3)| \geq 1/5 \).

7. Let \( f \) be analytic in \( \mathbb{D} \) and suppose \( |f(z)| < 1 \) on \( \mathbb{D} \). Let \( a = f(0) \). Show that \( f \) does not vanish in \( \{ z : |z| < |a| \} \).