1. Find the series expansion of
\[
\frac{z + 2i}{(z - 2)(z^2 + 1)}
\]
about the point 1.

2. (a) Suppose \( p \) is a polynomial with all its zeros in the upper half plane \( \mathbb{H} = \{ z : \text{Im} z > 0 \} \). Prove that all of the zeros of \( p' \) are contained in \( \mathbb{H} \). Hint: Look at the partial fraction expansion of \( p'/p \).

(b) Use (a) to prove that if \( p \) is a polynomial then the zeros of \( p' \) are contained in the (closed) convex hull of the zeros of \( p \). (The closed convex hull is the intersection of all half planes containing the zeros.)

3. Suppose \( f \) is analytic in a connected open set \( U \). If \( |f(z)| \) is constant on \( U \), prove that \( f \) is constant on \( U \). Likewise, prove that \( f \) is constant if \( \text{Re} f \) is constant.

4. Suppose \( f \) is analytic in a connected open set \( U \) such that for each \( z \in U \), there exists an \( n \) (depending upon \( z \)) such that \( f^{(n)}(z) = 0 \). Prove \( f \) is a polynomial.

5. Let \( f \) be analytic in a region \( U \) containing the point \( z = 0 \). Suppose \( |f(1/n)| < e^{-n} \) for \( n \geq n_0 \). Prove \( f(z) \equiv 0 \).

6. Suppose \( f \) has a power series expansion about 0 which converges in \( \mathbb{C} \) and suppose
\[
\int_{\mathbb{C}} |f(x + iy)| dx dy < \infty.
\]
Prove \( f \equiv 0 \).

7. (Challenge problem) (a) Define \( n^{-z} = e^{-z \ln(n)} \). Prove that
\[
\zeta(z) = \sum_{n=1}^{\infty} n^{-z}
\]
converges uniformly and absolutely in \( \{ z : \text{Re} z > c \} \) for \( c > 1 \).

(b) Show that
\[
\zeta(z) - \frac{1}{z - 1} = \sum_{n=1}^{\infty} n^{-z} - \int_{n}^{n+1} x^{-z} dx
\]
and show the sum converges uniformly and absolutely on compact subsets of \( \{ z : \Re z > 0 \} \). Also evaluate the integrals above.

Probably the most famous problem in all of mathematics is to prove that if \( 0 < \Re z < 1 \) and \( \zeta(z) = 0 \), then \( \Re z = 1/2 \).