Some are routine, but take time. A creative idea on others can lead to simple solutions. Learning how to write accurate, succinct and substantiated arguments is part of this course, so please write your solutions carefully. You may discuss the problems with others, but you must write the solutions in your own words.

1. Prove that $f$ has a power series expansion about $z_0$ with radius of convergence $r > 0$ if and only if $g(z) = \frac{f(z) - f(z_0)}{z - z_0}$ has a power series expansion about $z_0$, with the same radius of convergence. (How must you define $g(z_0)$, in terms of the coefficients of the series for $f$ to make this a true statement?)

2. For what values of $z$ is $\sum_{n=0}^{\infty} nz^n$ convergent? Same question for $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$.

Draw pictures for the regions.

3. Define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that this series converges for all $z$, that $e^z e^w = e^{z+w}$, that $e^{i\theta} = \cos \theta + i \sin \theta$ where $\cos \theta$ and $\sin \theta$ are defined by their series expansion as you learned them in calculus. Show $|e^z| = e^{Re z}$, and that $e^z = 1$ only when $z = 2\pi ki$ for some integer $k$.

4. Suppose $\sum_{j=0}^{\infty} |a_j|^2 < \infty$. Show $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{ z : |z| < 1 \}$. Compute

$$\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$  

(Prove your answer).

5. Prove the parallelogram equality:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

In geometric terms, the equality says that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the sides. It is perhaps a bit easier than a proof using high school geometry.
6. Suppose that $f$ is a continuous complex valued function on $[a, b]$. Let

$$A = \frac{1}{b-a} \int_a^b f(x) \, dx,$$

be the average of $f$ over the interval $[a, b]$.

(a) Show that if $|f(x)| \leq |A|$ for all $x \in [a, b]$, then $f$ is constant. (A picture might help)

(b) Show that if $|A| = (1/(b-a)) \int_a^b |f(x)| \, dx$, then $\arg f$ is constant.

Challenge problem: 7. Suppose $f$ is analytic in a convex open set $U$. Define $f'(\zeta)$ to be the coefficient of $z - \zeta$ in the power series expansion of $f$ based at $\zeta$ (hence $f'(\zeta) = \lim_{z \to \zeta} f(z)/z - f(\zeta)/z - \zeta$).

Suppose that for each $z, w \in U$ there exists a point $\zeta$ on the line segment between $z$ and $w$ with

$$\frac{f(z) - f(w)}{z - w} = f'(\zeta),$$

where $f'(\zeta)$ is defined to be the coefficient of $z - \zeta$ in the power series expansion about $\zeta$. Prove $f$ is a polynomial of degree at most 2. (The point is that you have to be careful: not all calculus theorems extend to similar complex versions.)

8. Stereographic projection combined with rigid motions of the unit sphere can be used to describe some transformations of the plane.

(a) Map $z \in \mathbb{C}$ to $S^2$, apply a rotation of the unit sphere, then map the resulting point back to the plane. For a fixed rotation, find this map of the extended plane to itself as an explicit function of $z$. Two cases are worth working out first: rotation about the $x_3$ axis and rotation about the $x_1$ axis.

(b) Another map can be obtained by mapping $z$ to $S^2$, then translating the sphere so that the origin is sent to $(x_0, y_0, z_0)$, then projecting back to the plane. The projection to the plane is given by drawing a line through the (translated) north pole and a point on the (translated) sphere and finding the intersection with the plane $\{(x, y, 0)\}$. For a fixed translation, find this map as an explicit function of $z$. In this case it is worth working out a vertical translation and a translation in the plane separately. Then view an arbitrary translation as a composition of these two maps.

Partial answer: the maps in part (a) and (b) are of the form $(az + b)/(cz + d)$ with $ad - bc \neq 0$

For a movie of these maps, see http://www.ima.umn.edu/arnold/moebius/index.html, but do the problem before viewing this link.