Project: Equilibrium Temperature Distributions

General Information. The flow of heat through a 2-dimensional object like a very thin metal plate can be determined by using a partial differential equation called the heat equation. If there are no external sources or sinks (places to lose heat) of heat, then eventually the temperature on the plate will remain constant at each point on the plate. In this case, the final (or equilibrium) temperature distribution is determined by Laplace's equation, which is also a partial differential equation.

More specifically, if the temperatures at the boundary of the plate are known, the resulting problem of finding the equilibrium temperature at each point on the plate is a specific example of something called the Dirichlet problem. The function modeling how the temperature is distributed is called a harmonic function for it satisfies the mean value property: the temperature at a given point must be the average of the temperatures at nearby points. Think about it; if this weren't true, heat would have to flow from or to that point from the surrounding points and the temperature wouldn't be in equilibrium.

Linear algebra can be used to approximate the equilibrium temperature distribution by replacing the partial derivatives with a "finite difference approximation" and applying a discrete version of the mean value property. Using something called Jacobi iteration allows one to approximate the final temperatures reached to as much accuracy as desired.

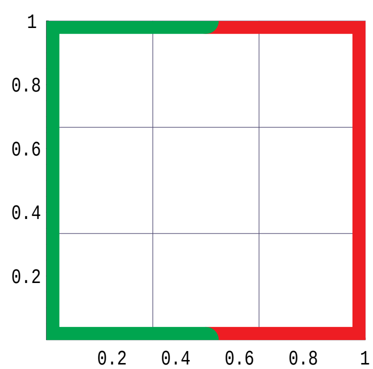
If you know about partial differential equations, you may find it interesting to research the heat equation, Laplace's equation, and the mean value property, but you need not do so. You will, however, need to learn about both of the mean value properties I mentioned above and how the discrete mean value property leads to systems of linear equations. The Jacobi iteration will also come up.

Related Words. Isotherms, Mesh points, Mean Value Property, Jacobi Iteration.

References. To learn more about the heat equation, Laplace's equation and the mean value property, it's best to look at a general book on Mathematical Physics or Partial Differential Equations. For information on the Jacobi iteration, you should probably consult a Numerical Analysis book.

Problems. For the following, consider a thin metal plate shaped like a unit square. Assume that the plate has boundary temperatures of $0°$ on exactly half of the circumference (left half) and $1°$ on the other (right) half.

1. A net with four interior mesh points is overlaid on the square as shown.



   (a) Using the discrete mean-value property, write the 4×4 linear system $\mathbf{t} = M\mathbf{t} + \mathbf{b}$ which determines the approximate temperatures at the four interior mesh points.

(b) Solve the linear system in (a).

(c) Use the Jacobi iteration scheme with an initial temperature of $t_i = 0°$ for each i to generate the first 5 iterations for the linear system in part (a). How far off is this from the solution found in (b)?

(d) By certain advanced methods, it can be determined that the exact temperatures at the four mesh points are $t_1 = t_3 = .2371$ and $t_2 = t_4 = .7629$. What are the percentage errors in the values found in part (b)?

(e) Find the sixth and seventh Jacobi iterations and determine how much, if any, you have reduced the percentage error.

2. Use the exact mean-value property to find the exact equilibrium temperature at the center of the plate.

3. Suppose the grid is refined to a grid with 9 interior mesh points. Set up the system of equations and calculate the first three Jacobi iterations if the initial temperature for each interior point is 1°.