Project: Electrical Networks

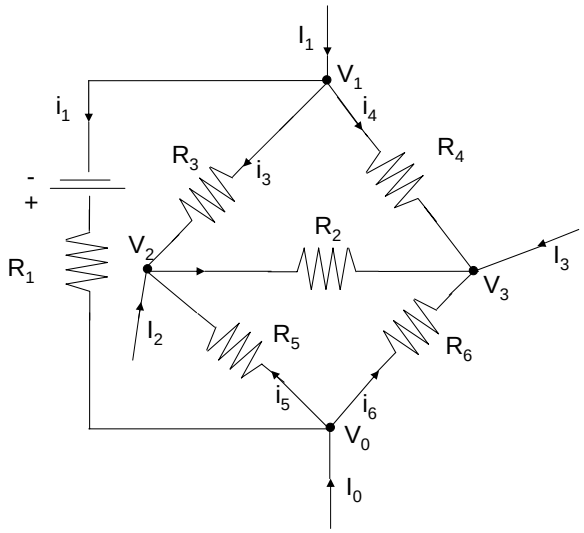
General Information. Electrical Networks are used in electrical engineering to model circuits. This project explores the application of linear algebra to electrical circuits, namely the linearity relationship between voltage and current. You will learn how to analyze moderately complicated circuits using linear algebra.

Key Words. Kirchoff's voltage and current laws, Ohm's law, The incidence matrix of a circuit, Wheatstone bridge.

References. No special tips for looking for references for this project.

Problems.

1. Consider the electrical network shown below.



The jagged lines correspond to resistors. The pair of parallel lines corresponds to a voltage source of e volts. Let I_0, I_1, I_2, I_3 denote the input currents at each of the nodes labeled 0 through 3, let $i_1, i_2, i_3, i_4, i_5, i_6$ be the currents through the various branches, as indicated, let V_0, V_1, V_2, V_3 be the voltages at each node and let $R_1, R_2, R_3, R_4, R_5, R_6$ be the resistance at each of the resistors in the diagram. Assume that $V_0 = 0$. Finally, we set

$$\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \qquad \mathbf{e} = \begin{bmatrix} e \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad R = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

(a) Find a matrix A such that $\mathbf{I} = A\mathbf{i}$.

(b) Show that $A^T\mathbf{V} = R\mathbf{i} - \mathbf{e}$.

(c) Use matrix operations to eliminate $\mathbf{i}$ from the above two equations and thus deduce that $AR^{-1}A^T\mathbf{V} = \mathbf{I} - AR^{-1}\mathbf{e}$. Show that, when written out, this equation becomes

$$\begin{bmatrix} Y_1+Y_3+Y_4 & -Y_3 & -Y_4 \\ -Y_3 & Y_2+Y_3+Y_5 & -Y_2 \\ -Y_4 & -Y_2 & Y_2+Y_4+Y_6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 - Y_1 e \\ I_2 \\ I_3 \end{bmatrix}$$

Here, $Y_i = \frac{1}{R_i}$.

(d) The system of linear equations in (c) represents the relationship between the input currents and the resulting voltages in the network. Show that this system of linear equations always has a unique solution. You need not give a formula for the solution.

2. Suppose that, in problem 1, the resistances are: $R_1 = 6$ ohms, $R_2 = 17$ ohms, $R_3 = 18$ ohms, $R_4 = 9$ ohms, $R_5 = 20$ ohms, $R_6 = 16$ ohms and $e = 15$ volts. Find the vector $\mathbf{V}$ in terms of $\mathbf{I}$.