Theorems of Fubini and Clairaut

In this note we'll prove that, for uniformly continuous functions on a rectangle, the Riemann integral is given by two iterated one variable integrals (Fubini) and as a Corollary, if f has mixed partials of order two which are continuous in a region, then the mixed partials are equal.

First we outline the existence of the Riemann integral with respect to area measure for a uniformly continuous function f. You might find it helpful to review our discussions of Riemann integrals on intervals in $\mathbb{R}$. Let $R = [a,b] \times [c,d]$ be a (finite) rectangle in $\mathbb{R}^2$. Let $a = x_0 < x_1 < \ldots < x_m = b$ be a partition of $[a,b]$ and let $c = y_0 < y_1 < \ldots < y_n = d$ be a partition of $[c,d]$. Let $\mathcal{P}$ be the corresponding partition of R into mn rectangles $R_{i,j} = [x_i, x_{i-1}] \times [y_j, y_{j-1}]$. Set $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$ and $m_{i,j} = \min_{R_{i,j}} f$ and $M_{i,j} = \max_{R_{i,j}} f$. Note that $\text{Area}(R_{i,j}) = \Delta x_i \Delta y_j$. Let $\delta(\mathcal{P}) \equiv \max \text{Area}(R_{i,j})$ be the mesh size of the partition $\mathcal{P}$. Let $L(f,\mathcal{P}) = \sum_{i=1}^m \sum_{j=1}^n m_{i,j} \Delta x_i \Delta y_j$ and let $U(f,\mathcal{P}) = \sum_{i=1}^m \sum_{j=1}^n M_{i,j} \Delta x_i \Delta y_j$ be the corresponding lower and upper Riemann sums. As in the one variable case, if $\mathcal{P}'$ is a refinement of the partition $\mathcal{P}$ (so each rectangle in $\mathcal{P}$ is a union of finitely many rectangles in the partition $\mathcal{P}'$) then

$$L(f,\mathcal{P}) \le L(f,\mathcal{P}')$$

because each $m_{i,j}$ is at most the minimum of f on each of the rectangles contained in $R_{i,j}$ in the partition $\mathcal{P}'$ and because the sum of the areas of those smaller rectangles equals the area of $R_{i,j}$. Similarly $U(f,\mathcal{P}') \le U(f,\mathcal{P})$. So each lower sum is bounded above by $U(f,\mathcal{P}_0)$ where $\mathcal{P}_0$ is the trival partition consisting of the single rectangle R. Moreover if $\mathcal{P}$ and $\mathcal{Q}$ are two partitions, there is a common refinement $\mathcal{S}$ with $L(f,\mathcal{P}) \le L(f,\mathcal{S})$ and $L(f,\mathcal{Q}) \le L(f,\mathcal{S})$. A similar statement is true for upper Riemann sums.

Note that by the uniform continuity of f, if $\varepsilon > 0$ then there is a $\delta_0 > 0$ so that if the mesh size of $\mathcal{P}$ satisfies $\delta(\mathcal{P}) < \delta_0$ then $M_{i,j} - m_{i,j} < \varepsilon$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ and hence

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) \le \sum_{i=1}^m \sum_{j=1}^n (M_{i,j} - m_{i,j}) \text{Area}(R_{i,j}) \le \varepsilon \text{Area}(R).$$

Since $\text{Area}(R)$ is finite, there is a unique number I so that the following limits exist and equal I:

$$\lim_{\delta(\mathcal{P}) \to 0} L(f,P) = \lim_{\delta(\mathcal{P}) \to 0} U(f,\mathcal{P}) = I$$

We define

$$\iint_R f(x,y) dA = I.$$

Fubini's theorem allows us to compute this integral using one variable integrals in two different ways.

Theorem (Fubini). *If f is uniformly continuous on a rectangle $R = [a, b] \times [c, d]$ then*

$$\iint\limits_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

The middle quantity in the statement of Fubini's theorem is found by first fixing x and integrating with respect to y. This new function of x is then integrated with respect to x. The third quantity in the statement of Fubini's theorem is similar, with the roles of x and y reversed.

Proof. Because $m_{i,j} \leq f(x,y) \leq M_{i,j}$ for all $(x,y) \in R_{i,j}$ we have that

$$m_{i,j}\Delta x_i \leq \int_{x_{i-1}}^{x_i} f(x,y)dx \leq M_{i,j}\Delta x_i$$

provided $y \in [y_j, y_{j-1}]$. Summing over all i we obtain

$$\sum_{i=1}^m m_{i,j}\Delta x_i \leq \int_a^b f(x,y)dx \leq \sum_{i=1}^m M_{i,j}\Delta x_i,$$

provided $y \in [y_j, y_{j-1}]$. Applying this comparison idea on each interval $[y_j, y_{j-1}]$ we obtain

$$\left(\sum_{i=1}^m m_{i,j}\Delta x_i \right) \Delta y_j \leq \int_{y_{j-1}}^{y_j} \int_a^b f(x,y)dxdy \leq \left(\sum_{i=1}^m M_{i,j}\Delta x_i \right) \Delta y_j.$$

Summing over j we obtain

$$\sum_{j=1}^n \sum_{i=1}^m m_{i,j}\Delta x_i \Delta y_j \leq \int_c^d \left(\int_a^b f(x,y)dx \right) dy \leq \sum_{j=1}^n \sum_{i=1}^m m_{i,j}\Delta x_i \Delta y_j.$$

We have shown now that for every partition $\mathcal{P}$ of the rectangle R

$$L(f,\mathcal{P}) \leq \int_c^d \left(\int_a^b f(x,y)dx \right) dy \leq U(f,\mathcal{P}).$$

But by our proof of the Riemann integrability of f, $\iint_R f(x,y)dA$ is the unique number which satisfies all the inequalities

$$L(f,\mathcal{P}) \leq \iint\limits_R f(x,y)dA \leq U(f,\mathcal{P}),$$

and so

$$\iint\limits_R f(x,y)dA = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

Switching the roles of x and y proves that the area integral is also equal to the iterated integral in the reverse order. □

We remark that a continuous function on a closed (finite) rectangle R is uniformly continuous on R. The proof is the same as the proof for closed (finite) intervals.

As a corollary we give a proof of Clairaut's theorem.

Theorem (Clairaut). *Suppose f is a differentiable function on an open set U in $\mathbb{R}^2$ and suppose that the mixed second partials f_{xy} and f_{yx} exist and are continuous on U. Then*

$$f_{xy} = f_{yx}.$$

Proof. We first note that if $R = [a,b] \times [c,d]$ is a rectangle contained in U then by Fubini's Theorem and the Fundamental Theorem of Calculus

$$\iint\limits_R (f_y)_x dA = \int_c^d \left(\int_a^b \frac{\partial (f_y(x,y))}{\partial x} dx \right) dy = \int_c^d (f_y(b,y) - f_y(a,y)) dy$$

$$= f(b,d) - f(b,c) - (f(a,d) - f(a,c)).$$

Similarly

$$\iint\limits_R (f_x)_y dA = \int_a^b \left(\int_c^d \frac{\partial (f_x(x,y))}{\partial y} dy \right) dx = \int_a^b (f_x(x,d) - f_x(x,c)) dx$$

$$= f(b,d) - f(a,d) - (f(b,c) - f(a,c)).$$

We conclude that

$$\iint\limits_R f_{yx} dA = \iint\limits_R f_{xy} dA.$$

We will prove Clairaut's theorem by contradiction. Suppose $f_{xy} - f_{yx} > 0$ at some point $(x_0, y_0) \in U$. Then because $f_{xy} - f_{yx}$ is continuous, there is a closed rectangle R contained in U so that $f_{xy} - f_{yx} > 0$ on all of R. But then $0 = \int_R (f_{xy} - f_{yx}) dA > 0$. A similar contradiction is obtained if $f_{xy} - f_{yx} < 0$ at some point $(x_1, y_1) \in U$. We conclude that $f_{xy} = f_{yx}$ at all points of U. $\square$