In this note we consider the problem of existence and uniqueness of solutions of the initial value problem

$$y' = f(t, y)\,, \qquad y(t_0) = y_0\,. \tag{1}$$

Suppose that $y = Y(t)$ is a solution defined for t near t_0. Then integrating both sides of (1) with respect to t gives

$$Y(t) - Y(t_0) = \int_{t_0}^{t} f(\tau, Y(\tau))\,d\tau$$

which we can rewrite in the form

$$Y(t) = y_0 + \int_{t_0}^{t} f(\tau, Y(\tau))\,d\tau \tag{2}$$

Notice that differentiating both sides of (2) with respect to t yields Equation (1). So Equation (2) is equivalent to the initial value problem (1).

Picard Iteration. Under certain conditions on f (to be discussed below), the solution of (2) is the limit of a Cauchy Sequence of functions:

$$Y(t) = \lim_{n\to\infty} Y_n(t)$$

where $Y_0(t) = y_0$ the constant function and

$$Y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, Y_n(\tau))\,d\tau \tag{3}$$

Example. Consider the initial value problem $y' = y$, $y(0) = 1$, whose solution is $y = e^t$ (using techniques we learned last quarter).

Substituting $f(t, y) = y$, $t_0 = 0$, and $y_0 = 1$ into (3) gives:

$$\begin{aligned}
Y_1(t) &= 1 + \int_0^t 1\,d\tau = 1 + t \\
Y_2(t) &= 1 + \int_0^t (1 + \tau)\,d\tau = 1 + t + t^2/2 \\
Y_3(t) &= 1 + \int_0^t (1 + \tau + \tau^2/2)\,d\tau = 1 + t + t^2/2 + t^3/6\,.
\end{aligned}$$

More generally, using Mathematical Induction, one can show that

$$Y_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!}\,.$$

Consequently,

$$\lim_{n\to\infty} Y_n(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t\,.$$

Conditions on the function $f(t, y)$. The initial value problem (1) does not always have a unique solution, for consider the initial value problem

$$\frac{dy}{dt} = f(y)\,, \quad y(0) = 0$$

where $f(y) = \begin{cases} 0 & \text{for } y \le 0 \\ \sqrt{2y} & \text{for } y \ge 0. \end{cases}$. Now for any $a > 0$, consider the function $\phi_a : \mathbb{R} \to \mathbb{R}$ defined as follows

$$\phi_a(t) = \begin{cases} (t-a)^2/2 & \text{for } t \ge a \\ 0 & \text{for } t \le a\,. \end{cases}$$

By construction, ϕ_a satisfies the initial condition $\phi_a(0) = 0$. It also satisfies the differential equation

$$\phi_a'(t) = f(\phi_a(t)) \text{ for all } t\,;$$

This is clear since

$$\phi_a'(t) = 0 = f(0) = f(\phi_a(t)) \text{ for } t \le a\,;$$

and

$$\frac{d(t-a)^2/2}{dt} = (t-a) = \sqrt{2(t-a)^2/2} = f((t-a)^2/2) \text{ for } t \ge a\,.$$

This example shows that we need to impose conditions on f if we want to ensure that (1) has a unique solution. Suppose that f satisfies the following condition:

Let R be the rectangular region

$$R = \{(t,y) \,:\, |t - t_0| \le a \text{ and } |y - y_0| \le b\}\,,\ \text{for } a, b > 0\,.$$

Then

(i) The function $f(t,y)$ is continuous as a function of t for all for all $(t,y) \in R$

(ii) There is a constant $K > 0$ such that f satisfies the inequality

$$|f(t,y) - f(t,z)| \le K\,|y - z|$$

for all (t,y) and (t,z) in R.

A function satisfying (ii) is said to be *Lipschitz continuous* with respect to y on R.

Theorem (Picard-Lindelöf). *Suppose f satisfies conditions (i) and (ii) above. Then for some $c > 0$, the initial value problem* (1) *has a unique solution $y = y(t)$ for $|t - t_0| < c$.*

We will prove the Picard-Lindelöf Theorem by showing that the sequence $Y_n(t)$ defined by Picard iteration is a Cauchy sequence of functions.

Set $M = Max_{(t,y)\in R}|f(t,y)|$ and set

$$c = \min\left(a, \frac{b}{M}, \frac{1}{2K}\right)\,,$$

and let $\mathcal{F}$ be the collection of all continuous functions $\phi : [t_0 - c, t_0 + c] \to \mathbb{R}$ defined as follows

$$\mathcal{F} = \{\phi : [t_0 - c, t_0 + c] \to \mathbb{R} \,:\, \phi(t_0) = y_0 \text{ and } |\phi(t) - y_0| \le b\}$$

Lemma. *1. Suppose that $\phi \in \mathcal{F}$. Then the function $\Phi = T[\phi]$ defined by*

$$\Phi(t) = y_0 + \int_{t_0}^{t} f(\tau, \phi(\tau))\, d\tau$$

is also in $\mathcal{F}$.

Proof. We first have to prove that Φ is well-defined. Set $g(t) = f(t, \phi(t))$. Then

$$\Phi(t) = y_0 + \int_{t_0}^{t} g(\tau)\, d\tau$$

If we can show that g is continuous, than it follows that the integral is well-defined. In fact, by the Fundamental Theorem of Calculus, it follows that Φ is differentiable, and therefore continuous.

Therefore, we have to show that g is continous. To show this, fix t in the interval $I = [t_0 - c, t_0 + c]$, and choose $\epsilon > 0$. Since f is continuous as a function of its first variable and ϕ is continuous, there is a $\delta > 0$ such that the both of the following conditions are satisfied for $s \in I$:

(i) If $|s - t| < \delta$, then $|f(s, \phi(t)) - f(t, \phi(t))| < \epsilon/2$

(ii) If $|s - t| < \delta$, then $|\phi(s) - \phi(t)| < \epsilon/(2K)$

Therefore, by the triangle inequality, if $|s - t| < \delta$, then

$$\begin{aligned} |g(s) - g(t)| = |f(s, \phi(s)) - f(t, \phi(t)| &= |f(s, \phi(s)) - f(s, \phi(t)) + f(s, \phi(t)) - f(t, \phi(t))| \\ &\le |f(s, \phi(s)) - f(s, \phi(t))| + |f(s, \phi(t)) - f(t, \phi(t))| \\ &\le K|\phi(s) - \phi(t)| + \epsilon/2 < K\epsilon/(2K) + \epsilon/2 = \epsilon\,. \end{aligned}$$

To see that Φ is in $\mathcal{F}$, note that by construction $\Phi(t_0) = y_0$. Finally notice that $|t - t_0| \le c \le a$ implies $|t - t_0| \le a$. So

$$\begin{aligned} |\Phi(t) - y_0| &\le \left| \int_{t_0}^{t} f(\tau, \phi(\tau))\, d\tau \right| \\ &\le M|t - t_0| \le Mc \le M(b/M) = b \end{aligned}$$

□

In light of the lemma we just proved, we may view Picard iteration as a map of the form

$$T : \mathcal{F} \to \mathcal{F}$$

Lemma. *T satisfies the condition*

$$\|T[\phi] - T[\psi]\| \le 1/2\|\phi - \psi\|,$$

for all ϕ, ψ in $\mathcal{F}$, where $\|\beta\| := \max_{|t-t_0|<c} |\beta(t)|$ for $\beta : [t_0 - c, t_0 + c \to \mathbb{R}$ continuous.

Proof. Suppose that ϕ and ψ are functions in $\mathcal{F}$, and compute as follows:

$$\begin{aligned}
|T[\phi](t) - T[\psi](t)| &= \left| \int_{t_0}^{t} f(\tau, \phi(\tau)) - f(\tau, \psi(\tau))\, d\tau \right| \\
&\leq K \left| \int_{t_0}^{t} \phi(\tau) - \psi(\tau)\, d\tau \right| \\
&\leq K \|\phi - \psi\| c \\
&\leq K \frac{1}{2K} \|\phi - \psi\| = 1/2 \|\phi - \psi\|
\end{aligned}$$

□

The Picard-Lindelöf Theorem follows from above lemma and Theorem 3 of the handout "Cauchy Sequences of Functions".