Uniqueness of the Laplace Transform

A natural question that arises when using the Laplace transform to solve differential equations is: Can two different functions have the same Laplace transform (in which case we could not distinguish these two functions by just looking at the Laplace transform).

A piecewise continuous function f is said to be of **exponential type** a, where a is a real number, if there is a constant $M < \infty$, so that

$$\left|\frac{f(t)}{e^{at}}\right| \leq M,$$

for all $t > N$. In other words, f doesn't grow faster that e^{at} in this sense. If f is a piecewise continuous function of exponential type a, then the Laplace transform $\mathcal{L}f(s)$ exists for $s > a$ (Exercise). As mentioned in class, we identify two piecewise continuous functions if they agree except possibly at the points of discontinuity.

Theorem. *Suppose f and g are piecewise continuous on $[0,\infty)$ and exponential type a. If $\mathcal{L}f(s) = \mathcal{L}g(s)$ for $s > a$ then $f(t) = g(t)$ for $t \geq 0$.*

Proof. If $\mathcal{L}f = \mathcal{L}g$ then $\mathcal{L}(f-g) = 0$. So it is enough to prove that if $\mathcal{L}f(s) = 0$ for $s > a$ then $f(t) = 0$ for all $t \geq 0$. Fix $s_0 > a$ and make the change of variables in the Laplace transform of $u = e^{-t}$. Then for $s = s_0 + n + 1$ we obtain

$$0 = \mathcal{L}f(s) = \int_0^\infty f(t)e^{-nt}e^{-s_0 t}e^{-t}dt = \int_0^1 u^n \left(u^{s_0} f(-\ln u)\right) du, \tag{1}$$

$n = 0, 1, 2, \ldots$ Let $h(u) = u^{s_0} f(-\ln u)$. Then h is piecewise continuous on $(0,1]$ and

$$\lim_{u\to 0} h(u) = \lim_{t\to\infty} e^{-s_0 t} f(t) = 0,$$

because $s_0 > a$. Thus if we define $h(0) = 0$, then h is piecewise continuous and satisfies

$$\int_0^1 h(u)p(u)du = 0,$$

for every polynomial p by (1). This implies that if g has a power series expansion which converges uniformly on $[0,1]$ then

$$\int_0^1 h(u)g(u)du = 0. \tag{2}$$

If h is not the zero function then replacing h with $-h$ if necessary, we can find a $u_0 \in (0,1)$ and an interval $J = [u_0 - c, u_0 + c] \subset [0,1]$ and an $c_1 > 0$ so that $h \geq c_1$ on J.

Consider the function $g(u) = \frac{1}{d}e^{-(\frac{u-u_0}{d})^2}$. If $d > 0$ then g has a power series expansion which converges uniformly on $[0,1]$, so that (2) holds.

Set

$$I_1 = \int_J g(u)du = \int_{u_0-c}^{u_0+c} g(u)du = \int_{-c/d}^{c/d} e^{-t^2}dt \tag{3}$$

and

$$I_2 = \int_{u_0+c}^{1} g(u)du = \int_{c/d}^{(1-u_0)/d} e^{-t^2}dt \tag{4}$$

and

$$I_3 = \int_{0}^{u_0-c} g(u)du = \int_{-u_0/d}^{-c/d} e^{-t^2}dt. \tag{5}$$

Set $A = \int_{-\infty}^{\infty} e^{-t^2}dt$. Then $A > 0$ and given $\varepsilon > 0$, there is a $\delta > 0$ so that if $0 < d \leq \delta$ then

$$I_1 \geq \frac{A}{2}, \quad 0 \leq I_2 \leq \varepsilon, \quad \text{and} \quad 0 \leq I_3 \leq \varepsilon.$$

Because $h \geq c_1 > 0$ on J and $|h| \leq N$, for some $N < \infty$,

$$\int_J h(u)g(u)du \geq c_1 A/2 > 0$$

and

$$\left| \int_{[0,1]\setminus J} h(u)g(u)du \right| \leq 2N\varepsilon.$$

and so

$$\int_0^1 h(u)g(u)du \geq c_1\frac{A}{2} - 2N\varepsilon > 0$$

provided $\varepsilon < \frac{c_1 A}{4N}$, contradicting (2). This proves that h is the zero function and so by the definition of f, we must have f equal to the zero function, proving the theorem. □

The idea for constructing the function g that violates (2), was to make it non-negative and essentially 0 off the interval J and have integral over J large, yet still be approximable by polynomials.

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