Uniqueness of the Laplace Transform

A natural question that arises when using the Laplace transform to solve differential equations is: Can two different functions have the same Laplace transform (in which case we could not distinguish these two functions by just looking at the Laplace transform).

A piecewise continuous function \( f \) is said to be of exponential type \( a \), where \( a \) is a real number, if there is a constant \( M < \infty \), so that

\[
\left| \frac{f(t)}{e^{at}} \right| \leq M,
\]

for all \( t > N \). In other words, \( f \) doesn’t grow faster that \( e^{at} \) in this sense. If \( f \) is a piecewise continuous function of exponential type \( a \), then the Laplace transform \( \mathcal{L}f(s) \) exists for \( s > a \) (Exercise). As mentioned in class, we identify two piecewise continuous functions if they agree except possibly at the points of discontinuity.

**Theorem.** Suppose \( f \) and \( g \) are piecewise continuous on \([0, \infty)\) and exponential type \( a \). If \( \mathcal{L}f(s) = \mathcal{L}g(s) \) for \( s > a \) then \( f(t) = g(t) \) for \( t \geq 0 \).

**Proof.** If \( \mathcal{L}f = \mathcal{L}g \) then \( \mathcal{L}(f - g) = 0 \). So it is enough to prove that if \( \mathcal{L}f(s) = 0 \) for \( s > a \) then \( f(t) = 0 \) for all \( t \geq 0 \). Fix \( s_0 > a \) and make the change of variables in the Laplace transform of \( u = e^{-t} \). Then for \( s = s_0 + n + 1 \) we obtain

\[
0 = \mathcal{L}f(s) = \int_0^\infty f(t)e^{-nt}e^{-s_0 t}e^{-t}dt = \int_0^1 u^n(u^{s_0}f(-\ln u))du, \tag{1}
\]

\( n = 0, 1, 2, \ldots \) Let \( h(u) = u^{s_0}f(-\ln u) \). Then \( h \) is piecewise continuous on \((0, 1]\) and

\[
\lim_{u \to 0} h(u) = \lim_{t \to \infty} e^{-s_0 t}f(t) = 0,
\]

because \( s_0 > a \). Thus if we define \( h(0) = 0 \), then \( h \) is piecewise continuous and satisfies

\[
\int_0^1 h(u)p(u)du = 0,
\]

for every polynomial \( p \) by (1). This implies that if \( g \) has a power series expansion which converges uniformly on \([0, 1]\) then

\[
\int_0^1 h(u)g(u)du = 0. \tag{2}
\]

If \( h \) is not the zero function then replacing \( h \) with \(-h\) if necessary, we can find a \( u_0 \in (0, 1) \) and an interval \( J = [u_0 - c, u_0 + c] \subset [0, 1] \) and an \( c_1 > 0 \) so that \( h \geq c_1 \) on \( J \).
Consider the function \( g(u) = \frac{1}{d} e^{-\left(\frac{u-u_0}{d}\right)^2} \). If \( d > 0 \) then \( g \) has a power series expansion which converges uniformly on \([0, 1]\), so that (2) holds.

Set
\[
I_1 = \int_J g(u)du = \int_{u_0+c}^{u_0+c} g(u)du = \int_{-c/d}^{c/d} e^{-t^2} dt
\]  
and
\[
I_2 = \int_{u_0+c}^1 g(u)du = \int_{c/d}^{(1-u_0)/d} e^{-t^2} dt
\]  
and
\[
I_3 = \int_0^{u_0-c} g(u)du = \int_{-c/d}^{u_0/d} e^{-t^2} dt.
\]

Set \( A = \int_{-\infty}^{\infty} e^{-t^2} dt \). Then \( A > 0 \) and given \( \varepsilon > 0 \), there is a \( \delta > 0 \) so that if \( 0 < d \leq \delta \) then
\[
I_1 \geq \frac{A}{2}, \quad 0 \leq I_2 \leq \varepsilon, \quad \text{and} \quad 0 \leq I_3 \leq \varepsilon.
\]

Because \( h \geq c_1 > 0 \) on \( J \) and \( |h| \leq N \), for some \( N < \infty \),
\[
\int_J h(u)g(u)du \geq c_1 A/2 > 0
\]
and
\[
\left| \int_{[0,1]\setminus J} h(u)g(u)du \right| \leq 2N\varepsilon.
\]
and so
\[
\int_0^1 h(u)g(u)du \geq c_1 A/2 - 2N\varepsilon > 0
\]
provided \( \varepsilon < \frac{c_1 A}{4N} \), contradicting (2). This proves that \( h \) is the zero function and so by the definition of \( f \), we must have \( f \) equal to the zero function, proving the theorem. \( \square \)

The idea for constructing the function \( g \) that violates (2), was to make it non-negative and essentially 0 off the interval \( J \) and have integral over \( J \) large, yet still be approximable by polynomials.