During class I was asked about for a criterion for convergence of Newton's method. Here is one relatively simple one.

Suppose we want to find c so that $f(c) = 0$. Let $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 0, 1, 2, \ldots$.

Proposition. *Suppose f and f' are continuous on $[a, c]$ with f' never equal to 0 on (a, c). Suppose also that for all $a < x < t < c$ we have that*

$$0 \leq \frac{f'(t)}{f'(x)} \leq 1. \tag{1}$$

Then for any $x_0 \in [a, c]$ we have that $x_0 \leq x_n \leq c$ and

$$\lim_{n \to \infty} x_n = c.$$

A similar result holds on an interval of the form $[c, b]$. Try to prove it yourself after reading the proof below.

proof. By the Mean Value Theorem, there is a t between x_0 and c so that

$$\frac{f(c) - f(x_0)}{c - x_0} = f'(t).$$

Applying (1) with $x = x_0$ and using $f(c) = 0$, we have that

$$0 \leq \frac{-f(x_0)}{(c - x_0)f'(x_0)} \leq 1.$$

Multiplying by $c - x_0$ and then adding x_0 we obtain

$$x_0 \leq x_0 - \frac{f(x_0)}{f'(x_0)} \leq x_0 + c - x_0 = c.$$

This implies $x_0 \leq x_1 \leq c$. But then the same conclusion holds if we had started with x_1 instead of x_0. So $x_1 \leq x_2 \leq c$. By induction x_n is an increasing sequence trapped between x_0 and c. So x_n has a limit, call it d. But then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ has the same limit. By continuity of f and f',

$$d = d - \frac{f(d)}{f'(d)},$$

and so we must have $f(d) = 0$, since $f'(d) \neq 0$. But f' is never zero in (a, c) so f is either increasing or decreasing. Because $f(c) = 0$, we must have $d = c$, which proves the Proposition.

As an application, if $f' > 0$ and $f'' < 0$ on $[a, c]$ then f' is decreasing and so (1) holds. Likewise if $f' < 0$ and $f'' > 0$ on $[a, c]$ then (1) holds. In particular if $f'(x)f''(x) < 0$, for $x \in [a, c]$ then (1) holds.

Try drawing a picture of the case when $f' > 0$ and $f'' < 0$ and you will see why $x_0 < x_1 < c$.