General Suggestions

(1) Know the definition of the definite integral, the definition of the indefinite integral, the definition of the inverse of a function.

(2) Know the precise statements of the important theorems. If you use them to prove something, be sure to verify the hypotheses of the theorems you use. Some of the "important theorems" are:

- Covered on last midterm, but you still need to remember them:

  Intermediate Value Theorem (B.1.2), Extreme Value Theorem (B.2.2), The Mean Value Theorem (4.1.1), The Chain Rule (3.5.6), Tests for local extrema (the first and second derivative tests).

- New this time:

  The Riemann Integrability Theorem (B.4.6), The Fundamental Theorem of Calculus (Theorems 5.3.5 and 5.4.2 – *be able to prove 5.3.5*), The Mean Value Theorem for Integrals (5.9.3), Continuity of the Inverse (B.3.1), Differentiability of the Inverse (B.3.2).

Specific Problems

1. Compute $f'(x)$ for

   (a) $f(x) = \arcsin\left(e^{\sin(x)}\right)$ (b) $f(x) = \displaystyle\int_{\ln x}^{x} \frac{dt}{1+t^6}$ (c) $f(x) = \dfrac{x^2\sin^3(x)}{\sin(2x)\cosh x}$

2. Evaluate each of the following integrals

   (a) $\displaystyle\int_0^1 \frac{x^3}{1+x^4}\,dx$ (b) $\displaystyle\int_0^1 \frac{\arctan(x)}{1+x^2}\,dx$ (c) $\displaystyle\int_2^5 \frac{1}{x^2-4x+13}\,dx$ (d) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

3. Let T be the triangle with vertices $(0,0)$, $(1,-1)$ and $(1,1)$. Determine the volume of the solid obtained by rotating T about the line $y=-x$.

4. Prove that $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$.

5. The base of a solid is the disk of radius $r>0$ bounded by the circle $x^2+y^2=r^2$. The cross sections of the solid perpendicular to the x-axis are squares. Find the volume of the solid.

6. Express the following as a definite integral and evaluate it:

$$\lim_{n\to\infty}\left\{\frac{e^{1/n}}{n}+\frac{e^{2/n}}{n}+\frac{e^{3/n}}{n}+\cdots+\frac{e^{(2n-1)/n}}{n}+\frac{e^{2n/n}}{n}\right\}$$

7. Compute $f'(x)$ for

$$\text{(a) } f(x) = x^{2x}e^{-x} \qquad \text{(b) } f(x) = \int_{\ln x}^{\cos 3x} \frac{dt}{t^8+1}$$

8. Let $f(x) = xe^{2-x}$. Find the critical points and inflection points of f, and sketch the graph, indicating any asymptotes.

9. Find the average value of $f(x) = \dfrac{\cos(x/2)}{3+\sin(x/2)}$ on the interval $[-\pi, \pi]$.

10. The region in the first quadrant bounded by the positive x-axis, the positive y-axis and the curve $y = 16 - x^4$ is revolved around the y-axis.

    (a) Set up the integral to find the volume of the solid by both the disc method and the shell method.

    (b) Find the volume by evaluating each of the integrals.

11. Let f be continuous on $(-\infty, \infty)$, and define F by $F(x) = \int_0^x f(s)\, ds$. Prove that $F'(x) = f(x)$ for all x. Start from the definition of derivative; do not simply invoke the Fundamental Theorem of Calculus. Be very clear about where you use the continuity of f.

12. The *error function* is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

    (a) Prove that $\operatorname{erf} : \mathbb{R} \to \mathbb{R}$ is a one-to-one, increasing, differentiable function.

    (b) Show that erf has a differentiable inverse.

    NOTE: *This is NOT an "epsilon-delta" problem. Use some of the theorems mentioned above.*

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous even function (i.e. $f(-x) = f(x)$). Prove that the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \int_0^x f(t)\, dt$$

    is a continuous (in fact differentiable) odd function (i.e. $F(-x) = -F(x)$). **Hint:** Look at page 283 of your text.

14. Let $0 < k < 1$ be a constant. The *Incomplete Elliptic Integral of the Second Kind* is the function $E_k : \mathbb{R} \to \mathbb{R}$ defined by the formula

$$E_k(x) = \int_0^x \sqrt{1 - k^2 \sin^2 s}\, ds$$

(a) Prove that $E_k(x)$ is differentiable.

(b) Prove that $E_k'(x) > 0$ for all x. What does this tell you about the inverse function $E_k^{-1}(x)$?

(c) Does the graph of E_k have any points of inflection? Explain.

15. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose further that there is a constant $K > 0$ such that

$$|f'(x)| \leq K \text{ for all } x \in (a, b)\,.$$

Recall that if $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of the interval $[a, b]$, then the upper and lower sums are given by

$$U_f(P) = \sum_{i=1}^{n} M_i \Delta x_i \text{ and } L_f(P) = \sum_{i=1}^{n} m_i \Delta x_i\,,$$

where $\Delta x_i = x_i - x_{i-1}$, $M_i = \max\limits_{x_{i-1} \leq x \leq x_i} f(x)$, and $m_i = \min\limits_{x_{i-1} \leq x \leq x_i} f(x)$.

Show that

$$U_f(P) - L_f(P) \leq (b - a)K\,\|P\|$$

where $\|P\| = \max_i \Delta x_i$.

Hint: *Use the Mean-Value Theorem to show that* $M_i - m_i \leq K\Delta x_i$.