Midterm 1 Math 134 Autumn 2015 Solutions

1. Suppose $f(x_1) = f(x_2) = f(x_3) = 0$ with $x_1 < x_2 < x_3$. By Rolle's Theorem (or the Mean Value Theorem) there are y_1 and y_2 with $f'(y_1) = f'(y_2) = 0$ and $x_1 < y_1 < x_2 < y_2 < x_3$. But $f'(x) = 2nx^{2n-1} + 2ax$ and $f''(x) = 2n(2n-1)x^{2n-2} + 2a$. Since $2n(2n-1) > 0$ and $x^{2n-2} \geq 0$, we conclude $f''(x) \geq 2a > 0$ for all x. But then f' is increasing: $f'(a) < f'(b)$ if $a < b$. This contradicts $f'(y_1) = f'(y_2) = 0$.

(There is another proof just examining the first derivative.)

2. One way to see the hint: If $|x| \leq |c|$ and $|y| \leq |d|$ then

$$\frac{x^4}{a} + \frac{y^4}{b} \leq \frac{c^4}{a} + \frac{d^4}{b}.$$

So if $(c, d) \in \mathcal{R}$ then the rectangle with vertices $(|c|, |d|)$, $(-|c|, |d|)$, $(-|c|, -|d|)$, $(|c|, -|d|)$ is also contained in $\mathcal{R}$. This rectangle has area $4|c||d|$. If

$$\frac{c^4}{a} + \frac{d^4}{b} < 1$$

then we can increase $|c|$ slightly (thereby increasing the area) and the inequality will still be true. Thus we may suppose $c \geq 0$, $d \geq 0$ and

$$\frac{c^4}{a} + \frac{d^4}{b} = 1.$$

In this case the area of the rectangle can be expressed as a function of c:

$$A(c) = 4c\left(b - \frac{bc^4}{a}\right)^{\frac{1}{4}}.$$

The function A is defined and continuous for $0 \leq c \leq a^{\frac{1}{4}}$, and differentiable on $(0, a^{\frac{1}{4}})$. By the Extreme Value Theorem, A has an absolute maximum on this interval. Moreover, the absolute maximum occurs at $c = 0$ or $c = a^{\frac{1}{4}}$ or at a critical point of A in $(0, a^{\frac{1}{4}})$. Differentiating A with respect to c we obtain:

$$A'(c) = 4\left(b - \frac{bc^4}{a}\right)^{\frac{1}{4}} + c\left(b - \frac{bc^4}{a}\right)^{-\frac{3}{4}}\left(\frac{-4bc^3}{a}\right).$$

Setting $A'(c) = 0$ we obtain

$$b - \frac{bc^4}{a} = c^4\frac{b}{a},$$

so that $c = (a/2)^{\frac{1}{4}}$. For this value of c we have $A(c) = 4(a/2)^{\frac{1}{4}}(b/2)^{\frac{1}{4}}$. Since $A(0) = 0$ and $A(a^{\frac{1}{4}}) = 0$, the maximum value of A must be $4(a/2)^{\frac{1}{4}}(b/2)^{\frac{1}{4}}$.

3. Let $L = lub\{x_n\}$. Because $x_n \leq M$ for all n, the lub exists (we proved this in class). By definition of lub, $x_n \leq L$ for all n. Given $\varepsilon > 0$ there exists an N so that $L - \varepsilon < x_N \leq L$, for otherwise $L - \varepsilon$ is a smaller upper bound. For $n \geq N$ it follows by induction that $x_N < x_n$. Thus

$$L - \varepsilon < x_N < x_n \leq L$$

for all $n \geq N$. This proves $\lim_{n \to \infty} x_n = L$.