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Complex Numbers

§1. Complex Numbers.

The "complex numbers" $\mathbb{C}$ consist of pairs of real numbers: $\{(x,y) : x,y \in \mathbb{R}\}$. The complex number (x,y) can be represented geometrically as point in the plane $\mathbb{R}^2$, or viewed as a vector whose tip has coordinates (x,y) and whose tail has coordinates $(0,0)$. The complex number (x,y) can be identified with another pair of real numbers (r,θ), called the polar coordinate representation. The line from $(0,0)$ to (x,y) has length r and forms an angle with the positive x-axis. The angle is measured by using the distance along the corresponding arc of the circle of radius 1 (centered at $(0,0)$). By similarity, the length of the subtended arc on the circle of radius r is $r\theta$.

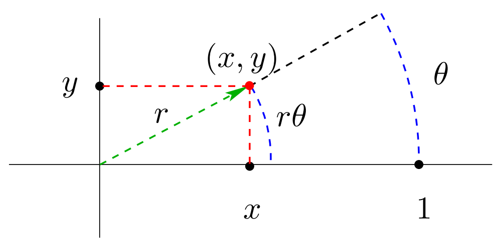

Figure I.1 Cartesian and polar representation of complex numbers.

Conversion between these two representations is given by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

and

$$r = \sqrt{x^2+y^2}, \qquad \tan\theta = \frac{y}{x}.$$

Care must be taken to find θ from the last equality since many angles can have the same tangent. However, consideration of the quadrant containing (x,y) will give a unique $\theta \in [0, 2\pi)$, provided $r > 0$ (we do not define θ when $r = 0$).

Addition of complex numbers is defined coordinatewise:

$$(a,b) + (c,d) = (a+c, b+d),$$

and can be visualized by vector addition.

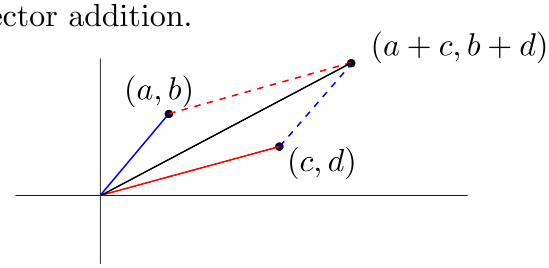

Figure I.2 Addition.

Multiplication is given by:

$$(a, b) \cdot (c, d) = (ac - bd, bc + ad)$$

and can be visualized as follows: The points $(0, 0), (1, 0), (a, b)$ form a triangle. Construct a similar triangle with corresponding points $(0, 0), (c, d), (x, y)$. Then it is an exercise in high school geometry to show that $(x, y) = (a, b) \cdot (c, d)$. By similarity, the length of the product is the product of the lengths and angles are added.

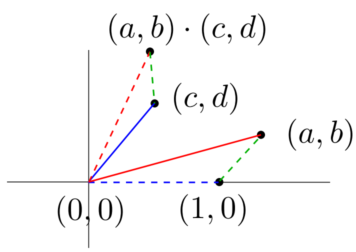

Figure I.3 Multiplication.

The real number t is identified with the complex number $(t, 0)$. With this identification, complex addition and multiplication is an extension of the usual addition and multiplication of real numbers. For conciseness when t is real, $t(x, y)$ means $(t, 0) \cdot (x, y) = (tx, ty)$. The additive identity is $0 = (0, 0)$ and $-(x, y) = (-x, -y)$. The multiplicative identity is $1 = (1, 0)$ and the multiplicative inverse of (x, y) is $(x/(x^2 + y^2), -y/(x^2 + y^2))$. It is a tedious exercise to check that the commutative and associative laws of addition and multiplication hold, as does the distributive law.

The notation for complex numbers becomes *much* easier if we use a single letter instead of a pair. It is traditional, at least among mathematicians, to use the letter i to denote the complex number $(0,1)$. If z is the complex number given by (x,y), then because $(x,y) = x(1,0) + y(0,1)$, we can write $z = x + yi$. If $z = x + iy$ then the "real" part of z is $\mathrm{Re} z = x$ and the "imaginary" part is $\mathrm{Im} z = y$. Note that $i \cdot i = -1$. We can now just use the usual algebraic rules for manipulating complex numbers together with the simplification $i^2 = -1$. For example, z/w means multiplication of z by the multiplicative inverse of w. To find the real and imaginary parts of the quotient, we use the analog of "rationalizing the denominator":

$$\frac{x+iy}{a+ib} = \frac{(x+iy)(a-ib)}{(a+ib)(a-ib)} = \frac{xa - i^2yb + iya - ixb}{a^2+b^2} = (\frac{xa+yb}{a^2+b^2}) + (\frac{ya-xb}{a^2+b^2})i.$$

Here is some additional notation: if $z = x + iy$ is given in polar coordinates by the pair (r, θ) then

$$|z| = r = \sqrt{x^2+y^2}$$

is called the modulus or absolute value of z. Note that $|z|$ is the distance from the complex number z to the origin 0. The angle θ is called the "argument" of z and written

$$\theta = \arg z.$$

The most common convention is that $-\pi < \arg z \leq \pi$, where positive angles are measured counter-clockwise and negative angles are measured clockwise. The complex conjugate of z is given by

$$\overline{z} = x - iy.$$

The complex conjugate is the reflection of z about the real line $\mathbb{R}$.

It is an easy exercise to show

$$
\begin{aligned}
|zw| &= |z||w| \\
|cz| &= c|z| \text{ if } c > 0, \\
z/|z| &\text{ has absolute value } 1, \\
z\overline{z} &= |z|^2, \\
\mathrm{Re} z &= (z + \overline{z})/2, \\
\mathrm{Im} z &= (z - \overline{z})/(2i), \\
\overline{z + w} &= \overline{z} + \overline{w}, \\
\overline{zw} &= \overline{z} \cdot \overline{w}, \\
\overline{\overline{z}} &= z, \\
|z| &= |\overline{z}|, \\
\arg zw &= \arg z + \arg w \quad \text{modulo } 2\pi, \\
\arg \overline{z} &= -\arg z = 2\pi - \arg z \quad \text{modulo } 2\pi.
\end{aligned}
$$

The statement "modulo 2π" means that the difference between the left and right hand sides of the equality is an integer multiple of 2π.

The identity $a + (z - a) = z$ expressed in vector form shows that $z - a$ is (a translate of) the vector from a to z. Thus $|z - a|$ is the length of the complex number $z - a$ but it is also equal to the distance from a to z. The circle centered at a with radius r is given by $\{z : |z - a| = r\}$ and the disk centered at a of radius r is given by $\{z : |z - a| < r\}$.

Complex numbers were around for at least 250 years before good applications were found. Cardano discussed them in his book Ars Magna (1545). Beginning in the 1800's, and continuing today, there has been an explosive growth in their usage. Now complex numbers are very important in the application of mathematics to engineering and physics.

It is a historical fiction that solutions to quadratic equations forced us to take complex numbers seriously. How to solve $x^2 = mx + c$ has been known for 2000 years and can be visualized as the points of intersection of the standard parabola $y = x^2$ and the line $y = mx + c$. As the line is shifted up or down by changing c, it is easy to see there are two, or one, or no (real) solutions. The solution to the cubic equation is where complex numbers really became important. A cubic equation can be put in the standard form

$$x^3 = 3px + 2q,$$

by scaling and translating. The solutions can be visualized as the intersection of the standard cubic $y = x^3$ and the line $y = 3px + 2q$. *Every* line meets the cubic, so there will always be a solution. By formal manipulations, Cardano showed that a solution is given by:

$$x = (q + \sqrt{q^2 - p^3})^{\frac{1}{3}} + (q - \sqrt{q^2 - p^3})^{\frac{1}{3}}.$$

Bombelli pointed out 30 years later that if $p = 5$ and $q = 2$ then $x = 4$ is a solution, but $q^2 - p^3 < 0$ so the above solution doesn't make sense. His "wild thought" was to use complex numbers to understand the solution:

$$x = (2 + 11i)^{\frac{1}{3}} + (2 - 11i)^{\frac{1}{3}}.$$

He found that $(2 \pm i)^3 = 2 \pm 11i$ and so the above solution actually equals 4. In other words, complex numbers were used to find a real solution. See Exercise 4.

Here are some elementary estimates which the reader should check:

$$-|z| \leq \text{Re}z \leq |z|$$

$$-|z| \leq \text{Im}z \leq |z|$$

and

$$|z| \leq |\text{Re}z| + |\text{Im}z|.$$

Perhaps the most useful inequality in analysis is the

Triangle inequality.

$$|z + w| \leq |z| + |w|,$$

and

$$|z + w| \geq \big||z| - |w|\big|.$$

The associated picture perhaps makes this result clear:

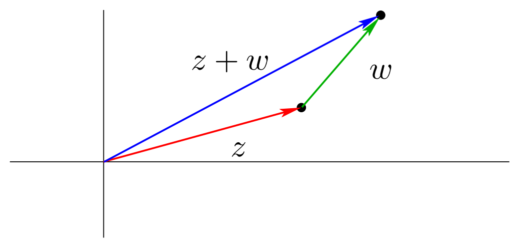

Figure I.4 Triangle inequality

Analysis is used to give a more rigorous proof of the triangle inequality (and it is good practice with the notation we've introduced):

Proof.

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) \\ &= z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w} \\ &= |z|^2 + 2\mathrm{Re}(w\overline{z}) + |w|^2 \\ &\leq |z|^2 + 2|w||\overline{z}| + |w|^2 \\ &= (|z|+|w|)^2. \end{aligned}$$

To obtain the second part of the triangle inequality:

$$|z| = |z+w+(-w)| \leq |z+w| + |-w| = |z+w| + |w|$$

and by subtracting $|w|$,

$$|z| - |w| \leq |z+w|,$$

and switching z and w:

$$|w| - |z| \leq |z+w|,$$

so that

$$\big||z| - |w|\big| \leq |z+w|. \qquad \square$$

§2. Polynomials.

We are interested in complex-valued functions of a complex variable. We could think of such functions in terms of real variables as maps from the plane $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f(x, y) = (u(x, y), v(x, y)),$$

and think of the graph of f as a subset of $\mathbb{R}^4$. But the subject becomes more tractable if we use a single letter z to denote in the independent variable and write $f(z)$ for the value at z, where $z = x + iy$ and $f(z) = u(z) + iv(z)$. For example

$$f(z) = z^n$$

is much simpler to write (and understand) than its real equivalent. Here z^n means the product of n copies of z.

The simplest functions are the **polynomials** in z:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n, \tag{1.1}$$

where $a_0, \ldots, a_n$ are complex numbers. If $a_n \neq 0$, then we say that n is the **degree** of p. Note that $\overline{z}$ is not a (complex) polynomial, and neither is $\mathrm{Re}z$ or $\mathrm{Im}z$.

Let's take a closer look at **linear** or degree 1 polynomials. For example if b is a (fixed) complex number, then

$$g(z) = z + b$$

translates, or shifts, the plane. If a is a (fixed) complex number then

$$h(z) = az$$

can be viewed as a dilation and rotation. To see this, observe that by Section 1 and Exercise 1, $|az| = |a||z|$ and $\arg(az) = \arg(a) + \arg(z)$ (up to a multiple of 2π). So that h dilates z by a factor of $|a|$ and rotates the point z by the angle $\arg a$. A linear function

$$f(z) = az + b$$

can then be viewed as a dilation and rotation followed by a translation. Equivalently, writing $f(z) = a(z + b/a)$ we can view f as a translation followed by a rotation and dilation.

Another instructive example is the function $p(z) = z^n$. By Section 1 again,

$$|p(z)| = |z|^n \quad \text{and} \quad \arg p(z) = n \arg z \mod 2\pi.$$

Each pie slice

$$S_k = \{z : \left|\arg z - \frac{2\pi k}{n}\right| < \frac{\pi}{n}\} \cap \{z : |z| < r\},$$

$k = 0, \ldots, n-1$ is mapped to a slit disk

$$\{z : |z| < r^n\} \setminus (-r^n, 0).$$

Angles between straight line segments issuing from the origin are multiplied by n and for small r, the size of the image disk is much smaller than the "radius" of the pie slice. See Figure I.4

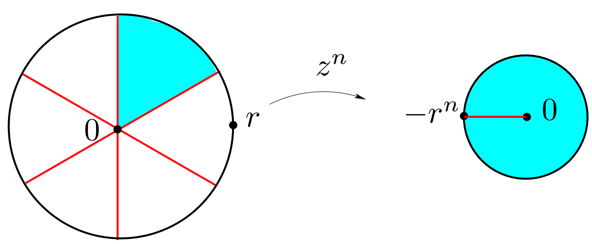

Figure I.4 The power map.

The function $k(z) = b(z - z_0)^n$ can be viewed as a translation by $-z_0$, followed by the power function, and then a rotation and dilation. To put it another way, k translates a neighborhood of z_0 to the origin, then acts like the power function z^n, followed by a dilation and rotation by b.

To understand the local behavior of a polynomial (1.1) near a point z_0, write $z = (z - z_0) + z_0$ and expand (1.1) by multiplying out and collecting terms to obtain:

$$p(z) = p(z_0) + b_1(z - z_0) + b_2(z - z_0)^2 + \ldots + b_n(z - z_0)^n. \tag{1.2}$$

Another way to see this is to note that $p(z) - a_n(z - z_0)^n$ is a polynomial of degree at most $n - 1$, so (1.2) follows by induction on the degree. If $b_1 \neq 0$ then $p(z)$ behaves like the linear function $p(z_0) + b_1(z - z_0)$ for z near z_0. If $b_1 = 0$ then near z_0, $p(z)$ is closely approximated by $p(z_0) + b_k(z - z_0)^k$, where b_k is the first non-zero coefficient in the expansion (1.2). Indeed, for small $\zeta = z - z_0$,

$$|p(z_0 + \zeta) - (p(z_0) + b_k\zeta^k)| \leq C|\zeta|^{k+1}, \tag{1.3}$$

for some constant C. Figure I.5 is sometimes called "walking the dog", where the walking path has radius $r = |b_k||\zeta|^k$ and the leash has length $s = C|\zeta|^{k+1}$. As ζ traces a circle centered at 0 of radius ε, the function $p(z_0)+b_k\zeta^k$ winds k times around the circle centered at $p(z_0)$ with radius r. For small ε, the function $p(z_0 + \zeta)$ also then traces a path which winds k times around $p(z_0)$, since $s < r$.

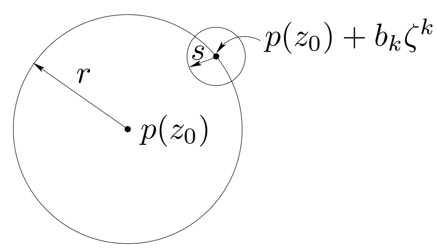

Figure I.5 $p(z_0 + \zeta)$ lies in a small disk of radius $s = C|\zeta|^{k+1} < r = |b_k||\zeta|^k$.

For z near z_0 then, $p(z)$ behaves like a translation by z_0, followed by a power function, a rotation and dilation, and finally a translation by $p(z_0)$.

Lemma 2.1. *Suppose p is a polynomial of degree n and suppose $B = \{z : |z - a| \leq r\}$ is a closed disk. Then*

(a) if $b \in \mathbb{C}$ then function

$$q(z) = \frac{p(z) - p(b)}{z - b}$$

is a polynomial of degree $n - 1$ and

(b) $|p|$ has a maximum and a minimum on B.

Proof. (a) Note that

$$z^n - b^n = (z - b)(b^{n-1} + zb^{n-2} + \ldots z^{n-2}b + z^{n-1}).$$

So if $p(z) = \sum_{k=0}^{n} a_k z^k$, then

$$q(z) = \frac{p(z) - p(b)}{z - b} = \sum_{k=1}^{n} a_k (\sum_{j=0}^{k-1} b^{k-1-j} z^j). \tag{2.1}$$

The coefficient of z^{n-1} in (2.1) is a_n so q is a polynomial of degree $n-1$.

(b) Suppose S is a square with $B \subset S$. Let $K = LUB\{|p(z)| : z \in B\}$. Suppose $z_n \in B$ and suppose $\lim_{n\to\infty} |p(z_n)| = K$. Divide S into 4 equal subsquares, at least one of which, call it S_1, must contain infinitely many z_n. Now divide S_1 into 4 equal subsquares, at least one of which, call it S_2, must contain infinitely many z_n. Repeat this process, obtaining a sequence of squares S_k with centers c_k such that $S_{k+1} \subset S_k$ and each S_k contains infinitely many z_n. Write $c_k = x_k + iy_k$. Because the size of S_k decreases to 0, we have $LUB\{x_k\} = GLB\{x_k\}$, so that $x = \lim x_k$ exists. Similarly $y = \lim y_k$ exists. Thus $c = x + iy = \lim c_k$ exists and is in B because $|z - a|$ is continuous. For each k, let w_k be one of the z_n in S_k. Then $\lim w_k = c$. Because sums and products of continuous functions are continuous, p and therefore $|p|$ is continuous. Thus $K = \lim |p(w_k)| = |p(c)|$. This proves that $|p|$ has a maximum in B. Replacing LUB with GLB, this same argument shows that $|p|$ has a minimum in B. $\square$

§3. The Fundamental Theorem of Algebra and Partial Fractions.

The "walking the dog" principle can be used to give a proof of an important result you've seen in some form or another since high school.

Theorem 3.1 (Fundamental Theorem of Algebra). *Every non-constant polynomial has a zero in* $\mathbb{C}$.

This remarkable result says that if we extend the real numbers to the complex numbers via the solution to the equation $z^2 + 1 = 0$ then every polynomial equation has a solution.

Proof. Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \quad n \geq 1$, is a polynomial which has no zeros and for which $a_n \neq 0$. Write

$$p(z) = z^n (a_n + a_{n-1}/z + a_{n-2}/z^2 + \ldots + a_0/z^n) = z^n h(z).$$

Because $|z^k| = |z|^k$, $h(z)$ converges to $a_n \neq 0$ and $|p(z)| \to \infty$ as $|z| \to \infty$. Thus for some R, $|p(z)| > |p(0)|$ for all $|z| > R$. By Lemma 2.1(b), $|p(z_0)| = \min_{\{z:|z|\leq R\}} |p(z)|$, for some $|z_0| \leq R$. Then $|p(z)| \geq |p(z_0)| > 0$ for all $z \in \mathbb{C}$. As in (1.2) we can write

$$p(z) = p(z_0) + b_k(z - z_0)^k + b_{k+1}(z - z_0)^{k+1} + \ldots + b_n(z - z_0)^n,$$

with $b_k \neq 0$. As in (1.4), if $|\zeta| = |z - z_0|$ is sufficiently small, then $r = |b_k\zeta^k| < |p(z_0)|$ and

$$|p(z_0 + \zeta) - (p(z_0) + b_k\zeta^k)| < C|\zeta^{k+1}| < r.$$

Moreover since $p(z_0) + b_k\zeta^k$ winds around the circle centered at $p(z_0)$ with radius r, we can choose ζ so that $|p(z_0) + b_k\zeta^k| = |p(z_0)| - r$. Then

$$|p(z_0 + \zeta)| \leq |p(z_0 + \zeta) - (p(z_0) + b_k\zeta^k)| + |p(z_0) + b_k\zeta^k| < r + |p(z_0)| - r = |p(z_0)|,$$

contradicting the minimality of $|p(z_0)|$. □

Corollary 3.2. *If p is a polynomial of degree n, then there are complex numbers $z_1, \ldots, z_n$ and a complex constant c so that*

$$p(z) = c\prod_{k=1}^{n}(z - z_k).$$

Corollary 3.2 does not tell us how to find the zeros, but it does say that there are exactly n zeros.

Proof. By Theorem 3.1, there is a complex number z_1 so that $p(z_1) = 0$. By Lemma 2.1(a) we can write $p(z) = (z - z_1)q(z)$ where q is a polynomial of degree $n - 1$. Again by Theorem 3.1, there is a complex number z_2 so that $q(z_2) = 0$, so by Lemma 2.1(a) we can write $q(z) = (z - z_2)r(z)$ where r is a polynomial of degree $n - 2$. Repeating this argument n times proves the Corollary. □

For example, the n distinct zeros of $z^n - 1$ are $\cos(2\pi k/n) + i\sin(2\pi k/n)$, $k = 0, 1, \ldots, n - 1$ which are equally spaced around the unit circle. See Exercise 3(c).

Recall that a rational function r is the ratio of two polynomials. By the Fundamental Theorem of Algebra we can write r in the form

$$r(z) = \frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}}.$$

The next Corollary, also probably familiar to you, allows us to write a rational function in a form that is easier to analyze. The form is also of practical importance because it allows us to solve certain differential equations that arise in Engineering problems using the Laplace transform and its inverse.

Corollary 3.3 (Partial Fraction Expansion). *If p is a polynomial then there is a polynomial q and constants $c_{k,j}$ so that*

$$\frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}} = q(z) + \sum_{j=1}^{N}\sum_{k=1}^{n_j} \frac{c_{k,j}}{(z - z_j)^k}. \tag{2.2}$$

Proof. There are two initial cases to consider: If p is a polynomial then

$$\frac{p(z)}{z - a} = q(z) + \frac{p(a)}{z - a}. \tag{2.3}$$

where $q(z) = (p(z) - p(a))/(z - a)$ is a polynomial, as in (2.1). Secondly, if $a \neq b$, we can write

$$\frac{1}{(z - a)(z - b)} = \frac{A}{z - a} + \frac{B}{z - b}, \tag{2.4}$$

for some constants A and B. For if this equation is true, then we can multiply each term on the right by $z - a$ and let $z \to a$ to obtain A on the right. The same process on the left yields $1/(a - b)$, and hence $A = 1/(a - b)$. Similarly $B = 1/(b - a)$. Now substitute these values for A and B into (2.4) and check that equality holds. The full theorem now follows by induction. Suppose the Corollary is true if the degree of the denominator is at most d. If we have an equation of the form (2.2) of degree d then we can divide each term in the equation by $z - a$. After division, the right side consists of lower degree terms to which the induction hypothesis applies, with one exception: when the denominator of the left side of (2.2) is $(z - b)^d$. If $a = b$, then after division by $z - b$, each term will be of the correct form. If $a \neq b$, then we could have applied the inductive assumption to the decomposition of

$$\frac{p(z)}{(z-b)^{d-1}(z-a)}$$

and then divided the result by $z-b$. $\square$

The proof above also suggests an algorithm for computing the coefficients $\{c_{k,j}\}$. First apply (2.3) with $a = z_1$. Multiply each term of the result by $1/(z-b)$ where b is one of the zeros of the denominator in (2.3) and apply either (2.3) or (2.4) to each of the resulting terms on the right side. Rinse and repeat. The algorithm can be speeded up because we know the form of the solution. For example if powers in the denominator n_j are all equal to one and if the numerator has smaller degree than the denominator, then the form is

$$\frac{p(z)}{\prod_{j=1}^{N}(z-z_j)} = \sum_{j=1}^{N} \frac{c_j}{z-z_j}. \tag{2.5}$$

If we multiply each term of the right side by $z-z_1$ then let $z \to z_1$, we obtain c_1. If we multiply the left side by the same factor, it cancels one of the terms in the denominator and letting $z \to z_1$ we obtain the value of the remaining part of the left side at z_1. This quickly gives c_1 and can be repeated for $c_2, \ldots, c_N$. This method is sometimes called the "cover-up method" because it can be done with less writing by observing that c_j is the value of the left side at z_j when you cover $z-z_j$ with your hand. If the denominator has terms with degree bigger than one, first use a denominator with all terms of degree one as above then as in the proof, multiply everything by $1/(z-b)$ and simplify all terms on the right, repeating as often as needed. If the degree of the numerator at any stage is not less than the degree of the denominator, use polynomial division to reduce the degree. Engineering problems typically have rational functions with real coefficients. See the Exercises for a similar technique that decomposes rational functions with real coefficients into terms whose denominators are either powers of linear terms with real zeros or powers of irreducible quadratics with real coefficients.

§4. Exercises.

1. Check the details of the high school geometry problem in the geometric version of complex multiplication.
2. Prove the parallelogram equality:

$$|z+w|^2+|z-w|^2=2(|z|^2+|w|^2).$$

   In geometric terms, the equality says that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the sides. It is perhaps a bit easier to prove it using the complex notation of this chapter than a proof using high school geometry.
3. (a) Suppose w is a non-zero complex number. Choose z so that $|z|=|w|^{\frac{1}{2}}$ and $\arg z=\frac{1}{2}\arg w$ or $\arg z=\frac{1}{2}\arg w+\pi$. Show that $z^2=w$ in both cases, and that these are the only solutions to $z^2=w$.
   (b) The quadratic formula gives two solutions to the equation $az^2+bz+c=0$, when a, b, c are complex numbers with $a\neq 0$ because completing the square is a purely algebraic manipulation of symbols, and there are two complex square roots of every non-zero complex number by part (a). Check the details.
   (c) Show that $z_k=\cos(2\pi k/n)+i\sin(2\pi k/n)$, $k=0,1,\ldots,n$ are the n distinct roots of $z^n-1=0$. Then use polar coordinates to find the n zeros of z^n-w where $w\neq 0$. Hint: Show $z_{k+1}=z_1z_k$.
4. Formally solve the cubic equation $ax^3+bx^2+cx+d=0$, where $x,a,b,c,d\in\mathbb{C}$, $a\neq 0$, by the following reduction process:
   (a) Set $x=u+t$ and choose the constant t so that the coefficient of u^2 is equal to zero.
   (b) If the coefficient of u is also zero, then take a cube root to solve. If the coefficient of u is non-zero, set $u=kv$ and choose the constant k so that $v^3=3v+r$, for some constant r.
   (c) Set $v=z+1/z$ and obtain a quadratic equation for z^3.
   (d) Use the quadratic formula to find two possible values for z^3, and then take a cube root to solve for z.

(e) Later we will see that the cubic equation has exactly three solutions, counting multiplicity. But the process in this exercise appears to generate more solutions, if we use two solutions to the quadratic and all three cube roots. Moreover there might be more than one valid choice for the constants used to reduce to a simpler equation. Explain.

5. (a) If p is a polynomial with real coefficients, prove that p can be factored into a product of linear and quadratic factors, each of which has real coefficients, and so that the quadratic factors are non-zero on $\mathbb{R}$. Most Engineering problems involving polynomials only need polynomials with real coefficients.

(b) For rational functions with real coefficients, such as those that typically occur in applications, it is sometimes preferable to use a partial fraction expansion without complex numbers in the expression. The cover-up method can also be used in this case. Here is an example to illustrate the idea. If a, b, c, d, and e are real show that

$$\frac{z^2 + dz + e}{(z-a)((z-b)^2 + c^2)} = \frac{A}{z-a} + \frac{B(z-b) + D}{(z-b)^2 + c^2},$$

where A, B, and D are real. Here we have completed the square for the irreducible quadratic factor. Notice also that we have written the numerator of the last term as $B(z-b) + D$, not $Bz + D$. We can find A by the usual cover-up method. Then to find B and D, we multiply by $(z-b)^2 + c^2$ and let it tend to 0. Thus $z \to b \pm ic$. Cover up the quadratic factor in the denominator on the left and let $z \to b + ic$. On the right side, when we multiply by the quadratic factor, the first term will tend to 0, the denominator of the second term will be cancelled and $B(z-b) + D$ tends to $Bic + D$. Thus the real part of the result on the left equals D and the imaginary part equals Bc, and then we can immediately write down the coefficients B and D. Try this process with two different irreducible quadratic factors in the denominator, and you'll see how much faster and accurate it is than solving many equations with many unknowns. The choice of the form of the numerator at $B(z-b) + D$ instead of $Bz + D$ made this computation a bit easier. It also turns out that it makes it a bit easier to compute inverse Laplace transforms of these rational functions, because the resulting term is a shift in the domain of a simpler function.