11/18

LAST TIME

L'HOSPITAL'S RULE

SUPPOSE f and g ARE DIFFERENTIABLE ON (a,b) and $g'(x) \neq 0$ on (a,b)

(1) $\lim_{x \to a^+} f(x) = 0$ (2) $\lim_{x \to a^+} g(x) = 0$ AND (3) $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$

THEN $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

- SAME HOLDS WITH a^+ REPLACED BY a^- OR JUST a.
- L CAN BE FINITE OR INFINITE
- CAN REPLACE (1) & (2) BY $\lim_{x \to a^+} f(x) = \infty$ or $-\infty$

  $\lim_{x \to a^+} g(x) = \infty$ or $-\infty$

- IF (1), (2), (3) DO NOT ALL HOLD, THEN WE CANNOT USE L'HOSPITAL

CAUCHY MEAN VALUE THEOREM

IF f & g ARE CONTINUOUS on $[a,b]$ AND DIFFERENTIABLE on (a,b) & if $g'(x) \neq 0$ on (a,b)

THEN THERE IS A NUMBER t in (a,b) so THAT

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(t)}{g'(t)}$$

PROOF:

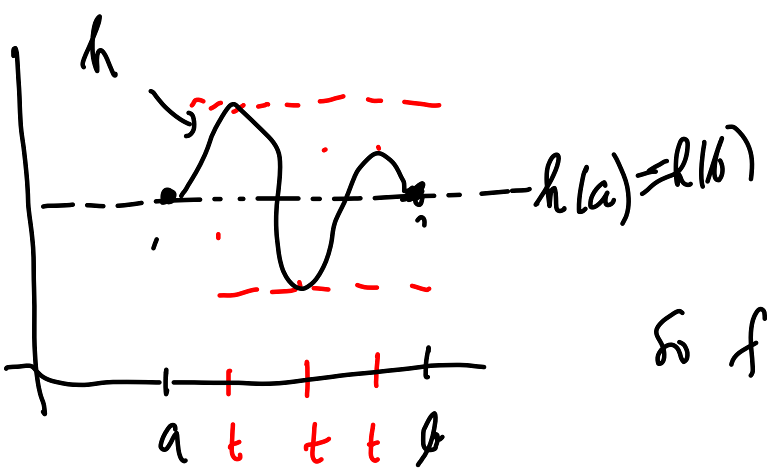
SET $h(x) = f(x) - r\,g(x)$

CHOOSE r SO THAT $h(a) = h(b)$

$$f(a) - r\,g(a) = f(b) - r\,g(b)$$

$$r\big(g(b) - g(a)\big) = f(b) - f(a).$$

$$r = \frac{f(b)-f(a)}{g(b)-g(a)}$$



SO THERE IS A t BETWEEN a & b WITH $h'(t) = 0$

So $f'(t) - r\,g'(t) = 0$

$$\frac{f'(t)}{g'(t)} = r = \frac{f(b)-f(a)}{g(b)-g(a)}$$

REMARK: $g(b) - g(a) \neq 0$ ELSE $g'(x)$ SOME x BETWEEN a & b $\boxtimes$

PROOF OF L'HOSPITAL:

SUPPOSE

1. $\lim_{x\to a^+} f(x) = 0$
2. $\lim_{x\to a^+} g(x) = 0$
3. $\lim_{t\to a^+} \frac{f'(t)}{g'(t)} = L$

f, g differentiable on (a,b)

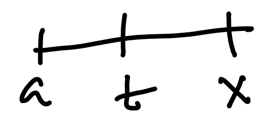
(and $g'(x) \neq 0$ on (a,b))

So f & g are continuous at a if we define $f(a)=g(a)=0$

THEN

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(t)}{g'(t)}$$

for some t



IF x is CLOSE TO a THEN t is CLOSE TO a, so by 3 THIS QUANTITY IS CLOSE TO L

So $\frac{f(x)}{g(x)} \to L$ as $x \to a^+$

(TRUE EVEN IF $L = \infty$ or $-\infty$)

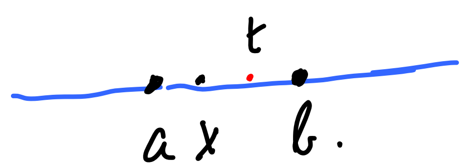
NOW REPLACE (1) & (2) by $1/f \to 0$ and $1/g \to 0$ as $x \to a$ if $b \neq a$.

THEN

$$\frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f(x)\left[1-\frac{f(b)}{f(x)}\right]}{g(x)\left[1-\frac{g(b)}{g(x)}\right]}$$

$$\| $$

$$\frac{f'(t)}{g'(t)}$$ FOR SOME t BETWEEN x & b.



FIX b CLOSE TO a THEN t IS CLOSE TO a

$\frac{f'(t)}{g'(t)}$ IS CLOSE TO L.

AS $x \to a$, t CHANGES, BUT STILL

AND $\frac{1-\frac{f(b)}{f(x)}}{1-\frac{g(b)}{g(x)}} \longrightarrow 1$ SO $\frac{f(x)}{g(x)} \longrightarrow L$

LAST CASE: WHAT ABOUT

$$\lim_{x\to\infty} \quad \text{or} \quad \lim_{x\to-\infty}$$

WRITE: $y = \frac{1}{x}$ then $y \to 0$ $\quad x = 1/y$

assume: $$\frac{f'(\frac{1}{y})}{g'(\frac{1}{y})} \longrightarrow L$$

$$\overset{\shortparallel}{\frac{f'(\frac{1}{y})(-1/y^2)}{g'(\frac{1}{y})(-1/y^2)}} = \frac{\frac{d}{dy} f(\frac{1}{y})}{\frac{d}{dy} g(\frac{1}{y})}$$

APPLY PREV. CASE

TO GET $\frac{f(1/y)}{g(1/y)} \longrightarrow L$ as $y \to 0$. So $\frac{f(x)}{g(x)} \to L$.

$(x = 1/y)$

$\boxtimes$

REVIEW; MIDTERM TUES: SEE PRACTICE EXAM

- PARAMETRIC EQUATIONS
- implicit DIFFERENTIATION
- LOGARITHMIC DIFFERENTIATION
- RELATED RATES
- LINEAR APPROXIMATION
- MAX/MIN/GRAPH PROBLEMS
- L'HOSPITAL'S RULE

EACH PROBLEM WILL COMBINE ONE OR MORE OF THESE.

Parametric CURVE

$(x(t), y(t))$

$\dot{x}$ = horizontal speed

$\dot{y}$ = vertical speed

$\dot{\ } = \frac{d}{dt}$

speed $= \sqrt{\dot{x}^2 + \dot{y}^2}$

slope of curve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$x = te^{-t^2}$

$y = \ln(1+e^t)$

FIND SPEED AT TIME t.

Implicit differentiation:

Given an equation involving x & y if possible, view y as a function of x, at least locally (replace variable y by $y(x)$)

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x \qquad \text{Find } \frac{dy}{dx}$$

LOGARITHMIC DIFF.

TAKE LOGS
SIMPLIFY
THEN DIFF.
SOLVE FOR y'

$$y = (\sin x)^{\tan x} (1+x)^{1/x}$$

$$\ln y = \tan x \ln(\sin x) + \frac{1}{x} \ln(1+x)$$

$$\frac{1}{y}\frac{dy}{dx} =$$

$$\frac{dy}{dx} =$$

Qeiw.

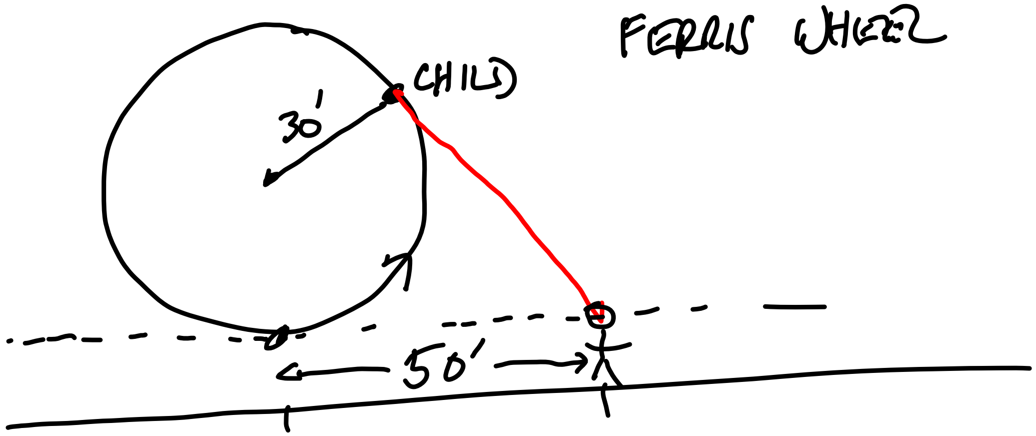
Related rates

In short:

- If you know the rate of change of a quantity and
- want the rate of change of another quantity
- Find an equation involving both quantities
- Then differentiate.

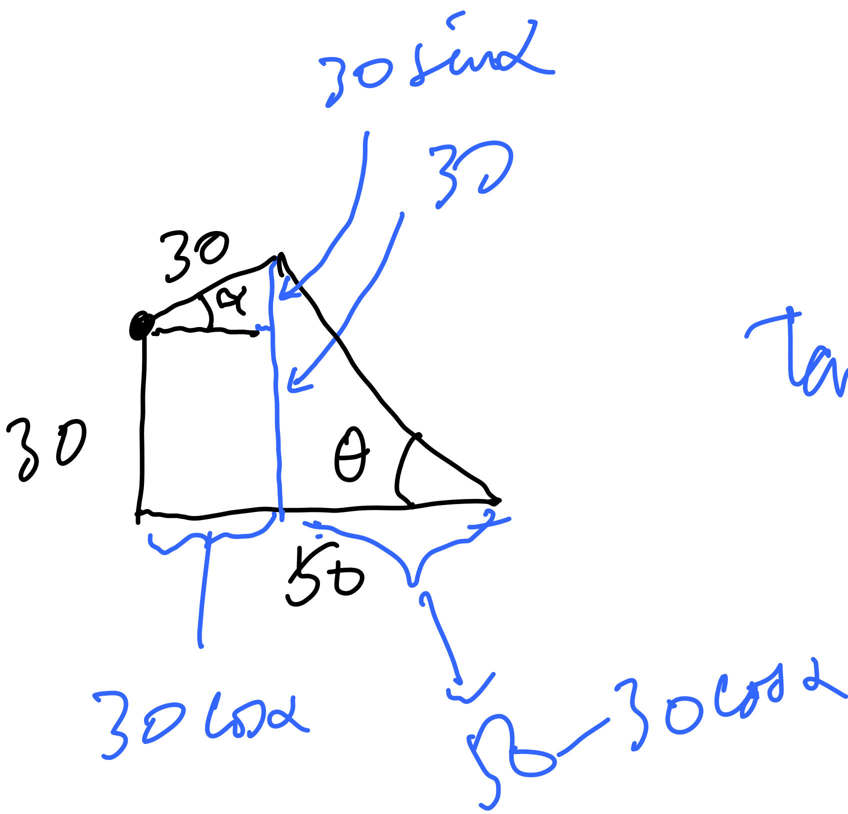
FERRIS WHEEL

PARAMETRIC RELATED RATES



5 REVOLUTIONS PER MINUTE

HOW FAST IS THE ANGLE OF INCLINATION OF THE PARENT'S EYES CHANGING WHEN THE CHILD IS AT THE TOP?



$$\frac{d\alpha}{dt} = \frac{5 \cdot 2\pi}{1} \ \frac{\text{rad}}{\text{min.}}$$

$$\tan\theta = \left[\frac{30\sin\alpha + 30}{50 - 30\cos\alpha}\right]$$

want $\frac{d\theta}{dt}$

Linear approximation of $f(x)$ at $x=a$ is

$$L(x) = f'(a)[x-a] + f(a)$$

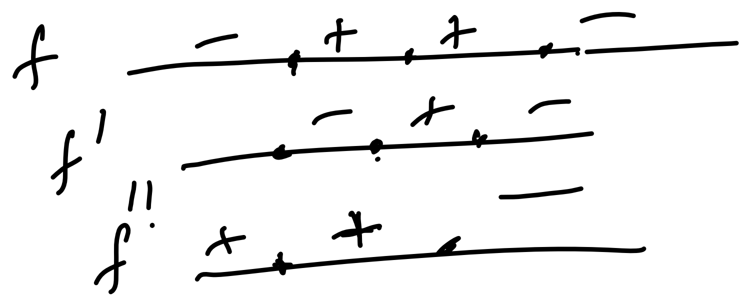
- IF YOU WANT TO KNOW SOMETHING ABOUT f NEAR $x=a$ BUT f IS TOO COMPLICATED, THEN
- ANSWER THE SAME QUESTION USING $L(x)$ INSTEAD IF $f(x)$

What is $\tan^{-1}(\sqrt{3} + .01)$ (approximately?)

MAX/MIN

GRAPH

SIGNS:



1. FIND WHERE $f = 0$ OR f UNDEFINED
2. FIND WHERE $f' = 0$ OR f' UNDEFINED
3. FIND WHERE $f'' = 0$ OR f'' UNDEFINED
4. FIND VALUES OF f OR LIMITS AT POINTS IN 1, 2, AND 3
5. FIND VERTICAL & HORIZONTAL ASYMPTOTES.
6. GRAPH. LABEL LOC MAX/MIN, INFLECT. PTS, CRITICAL NUMBERS/POINTS CONCAVITY

MIGHT NOT BE ABLE TO DO
ALL OF THE ABOVE IN SOME EXAMPLES
BUT DO AS MUCH AS POSSIBLE THEN
USE THE INFO TO GRAPH

GRAPH:

$$f(x) = \frac{x^2 - 3x}{(x+1)^2}$$

$$\lim_{h \to 0} \frac{\arctan(\sqrt{36-h}) - \arctan 6}{h}.$$

F(x).
Lin App
log diff

$$f(h) = \arctan(\sqrt{36-h})$$

$$f'(h) = \frac{1}{1+(\sqrt{36-h})^2} \frac{1}{2}(36-h)^{-1/2}(-1)$$

$$f'(0) = \frac{1}{1+(\sqrt{36})^2} \frac{1}{2}(36)^{-1/2}(-1)$$