11/16/16

ANNOUNCEMENTS:

OFFICE HOURS: TODAY 2–3

MON 1:30 – 2:20

WEDS 1:30 – 2:20

THEREAFTER: 2–3.

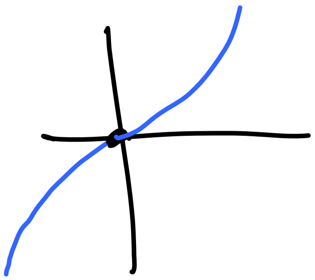
NOTE: 2 SECTIONS OF HW DUE FRI.

THURS LAST TIME TO ASK QUESTIONS IN QUIZ SECTION BEFORE MIDTERM

(YOU CAN, OF COURSE, ASK QUESTIONS IN LECTURE SECTIONS)

11/16 USING 1^{st} DERIV TO PROVE INEQUALITIES

where is. $\sin x \le x$?



$\boxed{f(x) = x - \sin x}$ when is $f \geq 0$

$f(0) = 0$

$f'(x) = 1 - \cos x \geq 0$ f increasing

$f(0) = 0$ so $f(x) \geq 0$ for $x > 0$

< 0 for $x < 0$

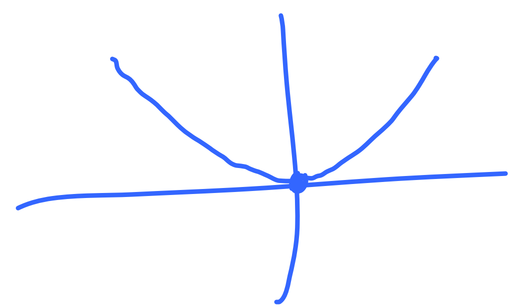
Is $\cos x \geq 1 - \frac{x^2}{2}$ for all x ?

let $f(x) = \cos x - (1 - x^2/2)$ $f(0) = 1 - (1 - 0) = 0$

$f'(x) = -\sin x + x = x - \sin x \geq 0$ for $x \geq 0$

f increasing for $x > 0$

So $f(x) \geq 0$ for $x \geq 0$



So $f(x) \geq 0$ for all x

yes.

SECT. 4.4

L'HOSPITAL'S RULE

L'Hospital

Suppose (1) $\lim_{x \to a} f(x) = 0$ and (2) $\lim_{x \to a} g(x) = 0$

and (3) Suppose $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ (SAME LIMIT)

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and ③ suppose $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ (SAME LIMIT)

COMMENTS:

- ALL THREE 1, 2, 3 MUST BE TRUE TO APPLY L'HOSPITAL'S RULE.
- L CAN BE FINITE OR $+\infty$ OR $-\infty$
- TRUE FOR $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$
- TRUE ALSO IF INSTEAD OF ① & ②, $1/f \to 0$ and $1/g \to 0$

  i.e. $f \to \pm\infty$
  $g \to \pm\infty$
- DO NOT USE $(f/g)'$

  NOT QUOTIENT RULE.

WHY IS L'HOSPITAL TRUE?

EASY CASE: IF f, g, f', g' CONTINUOUS AT a.

AND IF $f(a)=0$ AND $g(a)=0$

Then $\dfrac{f(x)}{g(x)} = \dfrac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} \longrightarrow \dfrac{f'(a)}{g'(a)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$

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INTERPRETATION: IF f & g ARE LINEAR
THEN $\frac{\text{NUM}}{\text{DENOM}} = \frac{\text{SLOPE OF GRAPH OF } f}{\text{SLOPE OF GRAPH OF } g}$

THIS IS NOT A PROOF OF L'HOSPITAL, IN FACT L'HOSPITAL IS MOST USEFUL WHEN THIS DOES NOT WORK.

WE'LL GIVE PROOF FRI
FIRST WE'LL GIVE SOME EXAMPLES

Examples:

$$\lim_{x \to 0} \frac{1-\cos x}{x^2}$$

3 THINGS TO CHECK: ① NUM $\to 0$

② DEN $\to 0$

③ RATIO OF DERIVATIVES $\to$

$$\frac{\sin x}{2x} \to \frac{1}{2}$$

④ CONCLUDE: By L'Hospital Rule:

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = 1/2$$

Examples:

$$\lim_{x \to \infty} \frac{x}{e^{2x}}$$

3 THINGS TO CHECK:

1. NUM $\to \infty$
2. DEN $\to \infty$
3. RATIO OF DERIVATIVES $\to$

(1) $\frac{1}{x} \to 0$

(2) $\frac{1}{e^{2x}} \to 0$

(3) $\frac{1}{2e^{2x}} \to 0$

By L'Hospital $\frac{x}{e^{2x}} \to 0$. (SAME LIMIT AS IN 3)

$$\lim_{x \to \infty} \frac{x^3}{e^{2x}} = 0$$ by L'Hosp.

$x^3 \to \infty$

$e^{2x} \to \infty$

$$\frac{3x^2}{2e^{2x}} \to 0$$ by L'H.

$3x^2 \to \infty$

$2e^{2x} \to \infty$

$$\frac{6x}{4e^{2x}} \to 0$$ by L'H.

$6x \to \infty$

$4e^{2x} \to \infty$

$$\frac{6}{8e^{2x}} \to 0$$

$$\frac{x^k}{e^x} \to 0 \text{ as } x \to \infty$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{?}{=} 0$$

$\ln x \to \infty$, $x \to \infty$ $\quad \infty \times \to \infty$

$\dfrac{\frac{1}{x}}{1} \to 0$

by L'Hosp.

$$\lim_{x\to 0^+} \frac{\ln x}{x} \overset{?}{=} -\infty$$

$\dfrac{\frac{1}{x}}{1} \to +\infty$ L'H does not apply

$$\lim_{x\to 0^+} x\ln x \overset{?}{=} 0$$ by L'Hospital.

$\dfrac{\ln x}{\frac{1}{x}}$

1. $\ln x \to -\infty$
2. $\frac{1}{x} \to +\infty$

$\dfrac{\frac{1}{x}}{-1/x^2} = -x$

3. $\to 0$

$x > 0$

$$\lim_{x \to \infty} x^{\frac{1}{x}}$$

Take ln first

$$\ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x} \ln x \longrightarrow 0$$

as $x \to \infty$

$$x^{\frac{1}{x}} = e^{\ln\left(x^{\frac{1}{x}}\right)} \longrightarrow e^{0} = 1$$

$$\lim_{x \to 0^+} x^{\frac{1}{x}} = 0$$

$$\ln\left(x^{\frac{1}{x}}\right) = \frac{\ln x}{x} \longrightarrow -\infty$$

$$x^{\frac{1}{x}} = e^{\ln\left(x^{1/x}\right)} \longrightarrow 0$$

$$\lim_{x \to 0} \underbrace{(1+x)^{1/x}}_{e^{\ln\left((1+x)^{\frac{1}{x}}\right)}} = e$$

$$\ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$

L'Hosp.

1 as $x \to 0$

$\ln(1+x) \to 0$

$x \to 0$

$$\frac{\frac{1}{1+x}}{1} \longrightarrow 1$$

$$\lim_{x\to 0} \frac{e^x-1}{x^2\sin\left(\frac{1}{x}\right)+2x}$$

$$\frac{e^x-1}{x}\underbrace{\left[x\sin\tfrac{1}{x}+2\right]}_{\to 2} \longrightarrow \frac{1}{2}$$

$e^x-1 \to 0,\ x \to 0$

L'Hop

$\to 1 \quad \frac{e^x}{1} \to e^0 = 1$

① $e^x-1 \longrightarrow 0$

② $x^2\sin\left(\frac{1}{x}\right)+2x \longrightarrow 0$

③ $$\frac{e^x}{2x\sin\left(\frac{1}{x}\right)+x^2\cos\left(\frac{1}{x}\right)\left[-\frac{1}{x^2}\right]+2}$$

$$= \frac{e^x}{2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)+2}$$

$e^x \to 1$

$2x\sin\left(\frac{1}{x}\right) \to 0$

$+2 \to 2$

NO LIMIT.