Solution to problem 1

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 2 & 14 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 8 \\ -2 \\ 14 \end{bmatrix}$$

Solution to problem 2

All solutions are found by reducing the matrix to echelon form and solving by back substitution.

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 8 \\ 0 & -3 & -1 & -2 \\ 2 & 14 & 4 & 14 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & 1 & 8 \\ 0 & -3 & -1 & -2 \\ 0 & -9 & -3 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & 1 & 8 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This reduced matrix corresponds to the linear system

$$\begin{cases} 2x_1 + 5x_2 + x_3 = 8 \\ \quad\; -3x_2 - x_3 = -2 \end{cases}$$

The system has two leading variables hence one free variable, so we set $x_3 = t$ and use back substitution to obtain the general solution, given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t/3 + 7/3 \\ -t/3 + 2/3 \\ t \end{bmatrix}$$

We can obtain any explict solution by specifying a value for t. Setting $t = 2$ yields the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

Solution to problem 3

The columns of A are not linearly independent. Recalling the definition of linear independence: the column vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of the matrix A are linearly independent if and only if the vector equation $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \mathbf{0}$ has only the trivial solution, $a_1 = a_2 = a_3 = 0$.
The above vector equation is a linear system of equations with augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 2 & 14 & 4 & 0 \end{array}\right]$$

By row reducing this matrix as we did above, we obtain a zero row, which gives us a free variable and infinitely many soutions to the homogeneous system in question. In particular this gives us non-trivial solutions, implying that the columns of A are linearly dependent.

Solution to problem 4

Observe that the big theorem applies to this situation (because we have 3 vectors in $\mathbb{R}^3$). The big theorem states (in part) that the given vectors are linearly independent if and only if they span $\mathbb{R}^3$. Since they are linearly dependent, the theorem implies that they do not span $\mathbb{R}^3$.

We can find the equation(s) of the subspace that they do span in several different ways. One way is to observe that $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution. By augmenting A with the arbitrary vector $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we row reduce to find

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & a \\ 0 & -3 & -1 & b \\ 2 & 14 & 4 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & 1 & a \\ 0 & -3 & -1 & b \\ 0 & -9 & -3 & a-c \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & 1 & a \\ 0 & -3 & -1 & b \\ 0 & 0 & 0 & a-c-3b \end{array}\right]$$

This means that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $a - c - 3b = 0$.
In other words, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if and only $a - c - 3b = 0$, thus

$$\text{span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 3y - z = 0 \right\}$$

which is the plane in $\mathbb{R}^3$ with equation $x - 3y - z = 0$.

Solution to problem 5

The big theorem answers this question completely since the columns of A are linearly independent if and only if they span $\mathbb{R}^3$ if and only if T is 1-1 if and only if T is onto. By way of the answers to problems 3 or 4, we conclude that T is neither 1-1 nor onto.

Solution to problem 6

(i) Recall that $\text{Ker}(T) = \left\{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}\right\}$. We found all such vectors by the method used in problem 3. Now we find a general form for them.

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 2 & 14 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -9 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Using back substitution, we obtain the general solution given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t/3 \\ -t/3 \\ t \end{bmatrix} = t \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \end{bmatrix}$.

This means that $\text{Ker}(T) = \text{span}\left\{ \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}$, hence a basis for $\text{Ker}(T)$ is $\left\{ \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}$.

(ii) There are multiple ways for finding a basis of $\text{Range}(T)$. One of the easier methods relies on the fact that the range of a linear transformation T is equal to the span of the columns of the matrix associated to T. If you don't remember why this fact is true you should make sure you can recall why (Hint: Look back at the original definition of the product $A\mathbf{x}$ which was defined in the lecture notes (page 6 of lecture 4 (section 2.2) page 6) on the course website).

By problem 4, the columns of A spanned the plane, P, given by the equation $x - 3y - z = 0$ so it remains to find a basis for this plane. Since it is given by one equation, we set y and z equal to free parameters $(y = t, z = s)$ and then solve for x. This gives the solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t + s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

thus $P = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Since these vectors are linearly independent, they form a basis for $P = \text{Range}(T)$.

(iii) The Range of T is equal to the span of the columns of A, which is equal to $\text{Col}(A)$ by definition, hence the basis obtained in the part (ii) is also a basis for $\text{Col}(A)$.

Solution to problem 7

Nullity(A)=dimension of Null(A)=dimension of Ker(T). Since the basis for Ker(T) contained only one vector, we conclude that nullity$(A) = 1$. The rank-nullity theorem implies that nullity(A)+rank(A)=3, hence rank$(A) = 2$. Recalling that rank$(A) =$ the dimension of Col(A), this agrees with the previous problem because Col(A) is a plane.

Solution to problem 8

One can directly compute this and should get an answer of 0. The computation-free approach follows from the big theorem since (among many other things) $\text{Ker}(T) = \mathbf{0}$ if and only if $\det(A) \neq 0$.

Solution to problem 9

Using the big theorem again, a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if it's associated matrix, A, is invertible if and only if $\det(A) \neq 0$. The matrix associated to S is BAB^{-1} and $\det(BAB^{-1})$ =$\det(B)\det(A)\det(B^{-1})$ =$\det(B)\det(A)\frac{1}{\det(B)}$=$\det(A) = 0$. This means that S is not invertible.

Solution to problem 10

To find the eigenvalues of A we find the roots of $\det\big(A - \lambda I_3\big)$ which is

$$\det\left(\begin{bmatrix} 2-\lambda & 5 & 1 \\ 0 & -3-\lambda & -1 \\ 2 & 14 & 4-\lambda \end{bmatrix}\right) = (2-\lambda)((3-\lambda)(4-\lambda)+14)+2(-2+\lambda) = -\lambda^3+3\lambda^2-2\lambda = 0$$

This factors as $-\lambda(\lambda^2-3\lambda+2) = -\lambda(\lambda-1)(\lambda-2)$ hence the eigenvalues are $0, 1$, and 2.

We find bases for the eigenspaces by finding bases for the subspaces $\mathrm{Null}(A)$, $\mathrm{Null}(A-I)$, and $\mathrm{Null}(A-2I)$ respectively, because E_λ =$\mathrm{Null}(A-\lambda I)$.

We already know E_0 =span$\left\{\begin{bmatrix} 1/3 \\ -1/3 \\ 1 \end{bmatrix}\right\}$ by problem 6 so it remains to find the others. Going through the same procedure as we did in problem 6, we find that

$$E_1 = \text{span}\left\{\begin{bmatrix} 1/4 \\ -1/4 \\ 1 \end{bmatrix}\right\} \text{ and } E_2 = \text{span}\left\{\begin{bmatrix} 2/5 \\ -1/5 \\ 1 \end{bmatrix}\right\}$$

Solution to problem 11

We can answer this question in multiple ways. An easy solution is that T is diagonalizable because we have distinct eigenvalues (hence an eigenbasis for $\mathbb{R}^3$). We diagonalize A with an invertible matrix P and diagonal matrix D satisfying $A = PDP^{-1}$ and here, the matrices are given by

$$P = \begin{bmatrix} 1/3 & 1/4 & 2/5 \\ -1/3 & -1/4 & -1/5 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$