Solutions to

Math 208(A) Final

December 11, 2023

NAME:

Section: 10:30 or 11:30 (circle the one you are registered in)

UW EMAIL:

Instructions. [1]

- **Please write your initials in the top right hand corner of each page.**

- There are 10 problems on this exam. Each one is worth 10 points for a total of 100 points. There is also one bonus problem at the end worth 5 points.
- For each problem below give a carefully explained solution using the vocabulary and notation from class. A correct answer with no supporting work or explanation will receive a zero.
- Simplify your answers, collect all terms, and reduce all fractions. Put a $\boxed{\text{box}}$ around your final answers.
- You are allowed a simple calculator and notesheet. Other notes, electronic devices, etc are not allowed. Take a few pencils from your pencil case out and put all other items away for the duration of the exam.
- All the questions can be solved using (at most) simple arithmetic. (If you find yourself doing complicated calculations, there might be an easier solution...)
- This exam is printed doubled sided. The last page is intentionally blank. You can use this as scratch paper or for more room for your solutions, but please label your work clearly if you intend for us to grade it.
- Raise your hand if you have any questions or spot a possible error.

Good luck!

[1] Test code: 3227

(1) Compute the determinant of the following matrix and briefly explain the steps you use for your computation. Put a box around the final answer.

$$\text{Det}\begin{bmatrix} 2 & 1 & 1 & 9 & -7 \\ 0 & 1 & -6 & 1 & 4 \\ 0 & 2 & 7 & 8 & -4 \\ 0 & 2 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} = 2(-4)\begin{vmatrix} 1 & -6 & 1 \\ 2 & 7 & 8 \\ 2 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -6 & 1 \\ 2 & 7 & 8 \\ 2 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} -6 & 1 \\ 7 & 8 \end{vmatrix} + 0 + 0$$ by cofactor expansion along bottom row

$$= 2(-48-7) = -110$$

$$\text{Answer} = -8 \times -110 = \boxed{880}$$

Note: other tests versions have diffent answers, but the block decomposition is the same.

Other answers: 848, 816, 800

(2) Find a 3×3 matrix A with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with $\lambda = 1$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ with $\lambda = 2$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with $\lambda = 10$.

From Ch 6 Conceptual problems.

(3) Find a maximal size independent subset of the following vectors and briefly decribe the method you used to verify your claim.

$$\begin{bmatrix}1\\0\\1\\7\end{bmatrix}, \begin{bmatrix}0\\1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\5\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\0\end{bmatrix}$$

← different on some tests so the determinant below may vary, but same 4 cols are independent in each case.

These vectors are in $\mathbb{R}^4$ so at most 4 of them can be independent.

$$\mathrm{Det}\begin{pmatrix}1&0&2&0\\0&1&0&0\\1&0&-1&0\\7&3&0&1\end{pmatrix} = \mathrm{Det}\begin{vmatrix}1&0&2\\0&1&0\\1&0&-1\end{vmatrix} = -1+2(-1) = -3$$

by cofactor expansion

so the column vectors in the matrix

$$\begin{pmatrix}1&0&2&0\\0&1&0&0\\1&0&-1&0\\7&3&0&1\end{pmatrix}$$

are a maximal size independent set of the given vectors.

(4) Give an example of a matrix that satisfies the following properties if possible or say "not possible" and give a brief justification. (2pts each)

(a) A 2×2 matrix that is nonzero in every entry and $A = A^{-1}$.

Let's try building one by choosing eigenvalues $-1, 1$ and eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$A^2 = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ so } A = A^{-1} \checkmark.$$

(b) A 2×2 matrix that is nonzero in every entry and $A = 2A$.

Not possible $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} \Rightarrow a=0, b=0, c=0, d=0$

so no such A exists.

(c) A 2×2 matrix that is nonzero in every entry and A and A^2 have different eigenvectors.

From (a): $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not eigenvector of A.

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ as eigenvectors.

(d) A 2×2 matrix that is nonzero in every entry and $A = A^T$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A^T$$

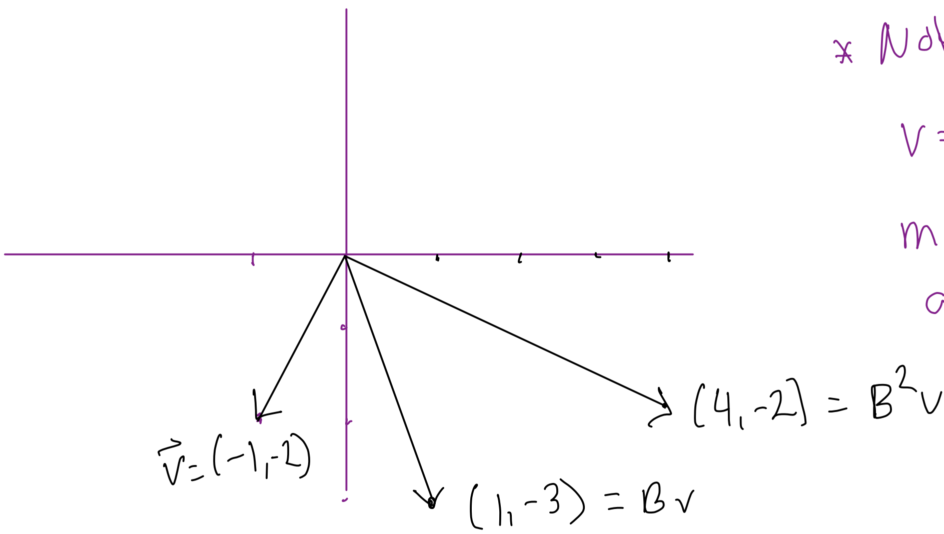
(e) A 2×2 matrix that is nonzero in every entry and $A^T = A^{-1}$.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = A^T \quad \checkmark$$

Check: $\begin{bmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(5) Let $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(a) Draw the three vectors $\mathbf{v}, B\mathbf{v}, B^2\mathbf{v}$ in $\mathbb{R}^2$ labeling the axes and the end-points of the vectors.



* Note, different tests had $v = \begin{bmatrix} \pm 1 \\ \pm 2 \end{bmatrix}$ so your picture might start with $\vec{v}$ in a different quadrant. But then Bv and $B^2\vec{v}$ rotate it by 45° and stretch it.

(b) Is there a value $k > 0$ such that $\mathbf{v}$ is an eigenvector of B^k? If so, what is the eigenvalue λ such that $B^k\mathbf{v} = \lambda\mathbf{v}$? If not, explain why not.

$$Bv = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\boxed{\text{Yes, } k = 4}$$

$$B^2 v = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

B rotates by 45° and scales vectors.

$$B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B^2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

So $B^3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = B\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$$B^4\begin{bmatrix} 1 \\ 0 \end{bmatrix} = B\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} = -4\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B^4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = B^3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = B^2\begin{bmatrix} -2 \\ 0 \end{bmatrix} = B\begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

So $B^4 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$ and $\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = -4\begin{bmatrix} -1 \\ -2 \end{bmatrix}$

So $\vec{v}$ is an eigenvector of B^4!

(6) Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 60 \\ 3 & -1 & 2 & 10 \\ 1 & 1 & -1 & 20 \end{bmatrix}.$$

(a) (2pts) What is the domain and codomain of the function $T(\mathbf{x}) = A\mathbf{x}$?

domain $= \mathbb{R}^4$ codomain $= \mathbb{R}^3$

since A is a 3×4 matrix

(b) (4pts) Give a basis for the range of T.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The cols of A span $\mathbb{R}^3$. $\operatorname{Det}\begin{pmatrix} 3 & 1 & 1 \\ 3 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \neq 0$

(c) (4pts) What is the dimension of the kernel of T? Justify your answer.

$1 = \dim(\ker(T))$. by rank-nullity thm.

$= \dim(\text{domain}(T)) - \dim(\text{range}(T))$

$= 4 - 3.$

(7) We have finally gotten data from our intergalactic frequency detector! The observations are

$$f(-1) = 8,\ f(0) = 4,\ f(1) = 4,\ f(2) = 2$$

It might be a polynomial function of the input. Please help us figure it out. Is there a cubic polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ that would match the data we have so far? If so, help us figure out the coefficients.

(a) (2pts) What equations must a_0, a_1, a_2, a_3 satisfy?

$a_0 - a_1 + a_2 - a_3 = 8$ since $f(-1)=8$

$a_0 = 4$ since $f(0) = 4$

$a_0 + a_1 + a_2 + a_3 = 4$ since $f(1) = 4$

$a_0 + 2a_1 + 4a_2 + 8a_3 = 2$ since $f(2) = 2$

(b) (6pts) Find all possible solutions to these equations.

Since $a_0 = 4$, let's eliminate that first + reduce to 3 equations.

$$\begin{aligned} -a_1 + a_2 - a_3 &= 4 \\ a_1 + a_2 + a_3 &= 0 \\ 2a_1 + 4a_2 + 8a_3 &= -2 \end{aligned}$$

Use the augmented matrix, after switch the 1st + 2nd equation

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 4 \\ 2 & 4 & 8 & -2 \end{array}\right) \underset{\substack{R_2+R_1 \to R_2 \\ R_3-2R_1 \to R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 2 & 6 & -2 \end{array}\right) \underset{R_3-R_2 \to R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 6 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

So $\boxed{a_0 = 4,\ a_1 = -1,\ a_2 = 2,\ a_3 = -1}$

(c) (2pts) Write down $f(x)$ and verify the data above is satisfied.

$f(x) = 4 - x + 2x^2 - x^3$

$f(-1) = 4 + 1 + 2 + 1 = 8$

$f(0) = 4$ ✓

$f(1) = 4 - 1 + 2 - 1 = 4$ ✓

$f(2) = 4 - 2 + 8 - 8 = 2$

(8) If C is the change of basis matrix that takes the basis $\mathcal{B}_1$ to $\mathcal{B}_2$ for $\mathbb{R}^n$, is C always, sometimes, or never invertible? Justify your answer.

always $C = B_2^{-1} B_1$ where

B_1 is formed by the colum vectors in $\mathcal{B}_1$ basis

B_2 has colums in $\mathcal{B}_2$ basis.

$\therefore$ B_1, B_2 are invertible $n \times n$ matrices since

$\mathcal{B}_1, \mathcal{B}_2$ are bases of $\mathbb{R}^n$.

The product of 2 invertible matrices is invertible.

Also $\text{Det}(B_2^{-1} B_1) = \text{Det}(B_2^{-1}) \cdot \text{Det}(B_1) \neq 0$

since $\text{Det}(B_2^{-1}) \neq 0$ and $\text{Det}(B_1) \neq 0$

(9) (a) Find 3×3 invertible matrices A and B such that $Det(A+B) = Det(A) + Det(B)$.

(b) For $n \times n$ matrices, does $Det(A+B) = Det(A) + Det(B)$ always hold? Justify your answer.

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad A+B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$Det(A) = 1$ $\quad Det(B) = -1$ $\quad Det(A+B) = 0$

$Det(A+B) = 0 = Det(A) + Det(B)$

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b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B$ $\quad A+B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$Det(A) = Det(B) = 1$ $\quad Det(A+B) = 8 \neq 1+1$.

(10) The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ is an infinite sequence given recursively by the formula $F_{n+1} = F_n + F_{n-1}$ and the initial conditions $F_1 = 1$ and $F_2 = 1$.

(a) (3pts) Find a matrix A such that $A^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and test it by computing $A\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then $A\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ by the formula $F_{n+1} = F_n + F_{n-1}$.

$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$,

$A^3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A \cdot A^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ etc.

Test: $A\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$ ✓

(b) (3pts) What are the eigenvalues of A?

$\mathrm{Det}(A - \lambda I) = \mathrm{Det}\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \mathrm{Det}\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)^2 - 1 = \lambda^2 - \lambda - 1$

Solving $\lambda^2 - \lambda - 1 = 0$ via the quadratic formula we get

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ so A has 2 distinct eigenvalues:

$\boxed{\frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}}$

(c) (2pts) What are the dimensions of the eigenspaces of A?

Since A is 2×2 and has 2 distinct eigenvalues we know the dimension of both eigenspaces is 1.

(d) (2pts) Is A diagonalizable, invertible, both, or neither?

$\boxed{\text{Both}}$

A is diagonalizable because it has 2 distinct eigenvalues $\Rightarrow \mathbb{R}^2$ has a basis of eigenvectors for A.

A is invertible because $\mathrm{Det}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$.

(11) (Bonus for 5pts) Let S be the set of all 5×5 matrices with entries in $\{0, 1\}$. What is the average of all determinants of matrices in S? (Hint: matrices with determinant 0 contribute 0 to the average.)

Answer: $\boxed{0}$

Why? Let $\mathrm{Mat}_5(0,1)$ = the set of all 5×5 matrices $A = (a_{ij})_{1 \le i,j \le 5}$ with each $a_{ij} \in \{0,1\}$.

There are 2^{25} matrices with entries in $\{0,1\}$.
To find the average, we compute

$$\text{Averge} = \frac{1}{2^{25}} \cdot \sum_{A \in \mathrm{Mat}_5(0,1)} \mathrm{Det}(A) = \frac{1}{2^{25}} \sum_{\substack{A \in \mathrm{Mat}_5(0,1) \\ \text{s.t. } \mathrm{Det}(A) \neq 0}} \mathrm{Det}(A)$$

since the matrices with $\mathrm{Det}(A) = 0$ won't change the sum.

By the Unifying Theorem, we know that

$\mathrm{Det}(A) \neq 0 \Leftrightarrow$ the rows of A are independent.

In particular, if two rows of A are the same, then $\mathrm{Det}(A) = 0$.

Case 1: If the first two rows of a 5×5 matrix A are the same then $\mathrm{Det}(A) = 0$ since its rows are dependent.

Case 2: If the first two rows of A are different, we can pair it with $A' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} A$ switching the top two rows. Then $\mathrm{Det}(A) + \mathrm{Det}(A') = 0$.

So
$$\text{average} = \frac{1}{2^{25}} \sum_{\substack{A \in \mathrm{Mat}_5(0,1) \\ \text{s.t. } \mathrm{Det}(A) \neq 0}} \mathrm{Det}(A) = \frac{1}{2^{25}} \left[\sum_{\substack{A \in \mathrm{Mat}_5(0,1) \\ \text{s.t. } \mathrm{Det}(A) > 0}} \mathrm{Det}(A) + \sum_{\substack{A \in \mathrm{Mat}_5(0,1) \\ \text{s.t. } \mathrm{Det}(A) < 0}} \mathrm{Det}(A) \right]$$

$= 0$ after cancelling all pairs. $\square$