STUDENT ID: ______________________________

Name______________________________

Final Exam Math 208D **Thursday, December 14, 2023, 2:30–4:20**

(5 questions for 100 points total)

Please read each problem carefully before beginning work, check your work as you go along, and show your work clearly.

1. (15 points) Let $A = \begin{bmatrix} 2 & 9 & 1 & 7 \\ 1 & 2 & -2 & 3 \\ -1 & -2 & 5 & 1 \\ 3 & 6 & -6 & 11 \end{bmatrix}$. **Find** $\det\left(\frac{1}{10}A^3\right)$. **Here** $\frac{1}{10}A^3$ **means scalar multiplication by** $\frac{1}{10}$ **times the matrix** A^3**.**

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2. (20 points) The matrix

$$A = \frac{1}{9}\begin{bmatrix} 1 & 8 & -4 \\ 8 & 1 & 4 \\ -4 & 4 & 7 \end{bmatrix}$$

is the matrix of a linear transformation that reflects vectors through the plane P. Find an equation for P of the form $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

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3. (15 points) A linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ with matrix A has the two properties: (1) its kernel consists of the line through the origin with slope 2, and (2) any vector perpendicular to this line is a fixed vector of T.

(a) Diagonalize A, that is, write it in the form PDP^{-1}. You do <u>not</u> have to multiply out the three matrices P, D, and P^{-1} (but the matrix P^{-1} needs to be computed).

(b) Let $B = A + I_2$, and similarly write B^4 in diagonalized form as the product of three matrices.

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4. (25 points) A 5×5 **matrix** A **with 3-dimensional nullspace has characteristic polynomial whose graph is given at the right:**

If "not enough information" applies to any of the following, then that can be your answer. These are short answer questions; no justification is needed.

(a) (8 points) Write out the characteristic polynomial in factored form.

(b) (4 points) nullity$(A - I) =$ ___ **and det**$(A - I) =$ ___

(c) (4 points) nullity$(A + I) =$ ___ **and det**$(A + I) =$ ___

(d) (3 points) Do the columns of A **span** $\mathbb{R}^5$**?**

(e) (3 points) Do the eigenvectors of A **span** $\mathbb{R}^5$**?**

(f) (3 points) Is A **diagonalizable?**

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5. (25 points) A function of the form

$$f(t) = x_1 \sin(t) + x_2 \cos(t) + t\Big(x_3 \sin(t) + x_4 \cos(t)\Big),$$

where $\overline{x}$ is the vector of coefficients, has derivative of the same form

$$f'(t) = y_1 \sin(t) + y_2 \cos(t) + t\Big(y_3 \sin(t) + y_4 \cos(t)\Big),$$

with coefficient vector $\overline{y} = A\overline{x}$.

(a) Find the matrix A.

(b) Find A^2.

(c) Find the characteristic polynomial of A^2.

(d) Find the eigenvalue(s) of A^2.

(e) Find the eigenspace of an eigenvalue λ in part (d) (of any one of them, if there's more than one eigenvalue). Do this by finding the solution space of the linear system $(A^2 - \lambda I_4) = 0$, written in terms of parameter(s) and basis vector(s).

(f) Which functions $f(t)$ are eigenfunctions for the second derivative? What fact from calculus would allow you to conclude that they're eigenfunctions for A^2 without going through parts (a)-(e)?

(CONTINUE TO PAGE 6 FOR MORE SPACE TO SHOW YOUR WORK ON THIS PROBLEM)

(continuation of your work on #5)

Math 208D Final Exam — Answers

1. (This is similar to #1 and #2 in the Chapter 5 conceptual problems.) If we switch the first and second rows and then subtract or add suitable multiples of the first row from the other rows so that the first column under the pivotal 1 has only zeros, we find that A has determinant

$$-\det\left(\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}\right) = -5 \cdot 3 \cdot 2 = -30.$$

Then $\det\left(\frac{1}{10}A^3\right) = 10^{-4}(-30)^3 = -2.7.$

2. (This is similar to #1 in the Chapter 4 conceptual problems and #4 in the Chapter 6 conceptual problems.) The coefficient vector $\overline{a}$ for the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$ of the plane P is perpendicular to the plane P, and so the reflection takes that vector to its negative. This means that $[a_1\ a_2\ a_3]^T$ is a solution to $(A+I_3)\overline{a} = 0$. After a few row operations we find the reduced-echelon form of $A + I_3$:

$$A + I_3 = \frac{1}{9}\begin{bmatrix} 10 & 8 & -4 \\ 8 & 10 & 4 \\ -4 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} 10 & 8 & -4 \\ 8 & 10 & 4 \\ 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & 18 & 36 \\ 0 & 18 & 36 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remember that our unknowns here are a_1, a_2, a_3. The free variable is $a_3 = s$, and then the two nonzero rows give us $a_1 = 2s$ and $a_2 = -2s$. We can choose $s = 1$ for simplicity (since we need only one normal vector to the plane in order to write the plane's equation). Thus, we take $[a_1\ a_2\ a_3]^T = [2\ \ -2\ 1]^T$, and the equation of the plane of reflection is $2x_1 - 2x_2 + x_3 = 0$.

3. (This is similar to #3 in the Chapter 6 conceptual problems.) (a) Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a vector on the line with slope 2 (and so is an eigenvector for eigenvalue 0), and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a perpendicular vector (and hence a fixed vector), we can write

$$A = PDP^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}.$$

(b) $(A + I_2)^4 = P(D + I_2)^4P^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}.$

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4. (This is similar to #7 in the Chapter 6 conceptual problems.)

(a) $\det(A - \lambda I_5) = (3 - \lambda)(0 - \lambda)^3(-1 - \lambda) = -\lambda^3(\lambda + 1)(\lambda - 3)$**.**

(b) nullity=0; set $\lambda = 1$ in part (a): determinant=$-2(-2) = 4$. (c) nullity=1; determinant=0; (d) no; (e) yes; (f) yes (NOTE: in (f) a correct answer means the same answer as in part (e)).

5. (This is similar to #8 in the Chapter 5 conceptual problems and #9(b) in the Chapter 6 conceptual problems.)

$$(\mathbf{a})\ A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (\mathbf{b})\ A^2 = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(c) $(\lambda+1)^4$**. (d)** $\lambda = -1$**. (e)** $A^2 - \lambda I_4 = A^2 + I_4 = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(in reduced echelon form). The solution space consists of $\overline{x} = s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

for any values of the parameters s_1 and s_2.

(f) $f(t) = x_1 \sin(t) + x_2 \cos(t)$ **for any values of x_1, x_2. The calculus fact is that** $\frac{d^2}{dt^2} \sin(t) = -\sin(t)$ **and the same for** $\cos(t)$**.**