Name: ____________________

- Do not open this exam until you are told to begin. You will have one hour and fifty minutes.
- Unless the problem says otherwise, **show your work** and explain your reasoning.
- If there is no work supporting an answer (even if the answer is correct) you will not receive full credit for the problem.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- You may freely refer to any theorems from class or your textbook.
- You are allowed a two-sided, handwritten sheet of notes and a TI-30X IIS calculator.
- If you have a question during the exam, raise your hand.
- **Do not write within 1cm of the edge!** Your exam will be scanned for grading.
- Draw a box around your final answer to each problem.
- At the start, I recommend you flip through the exam and glance at each problem. Good luck!

Question	Points	Score
1	32	
2	10	
3	5	
4	5	
5	5	
6	15	
7	8	
8	10	
9	10	
Total:	100	

Ethics statement: I know that my integrity is worth more than any exam grade. I pledge that I will neither give nor receive any unauthorized help on this exam.

Signature: ____________________

1. (32 points) For each situation below, either **give an example** (if possible) or **explain why it is not possible.** (If you can give an example, no further justification is necessary.)

   (a) A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ that spans a subspace S with $\dim(S) < k$.

   (b) A 3×3 matrix A and a vector $\mathbf{c} \in \mathbb{R}^3$ satisfying the following properties: (1) $A\mathbf{x} = \mathbf{c}$ has infinitely many solutions and (2) for every $\mathbf{b} \in \mathbb{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

   (c) A matrix A that is invertible but its inverse A^{-1} is not invertible.

   (d) An invertible matrix A and a nonzero matrix B such that AB is not invertible.

For each situation below, either **give an example** (if possible) or **explain why it is not possible.** (If you can give an example, no further justification is necessary.)

(e) A 3×3 matrix A with $\det(A) = 0$ and all of the entries of A are nonzero.

(f) A 2×2 matrix A with determinant 2 such that $A^T A = I_2$.

(g) A vector $\mathbf{x} \in \mathbb{R}^2$ and two different bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$ such that $[\mathbf{x}]_{\mathcal{B}}$ is the same as $[\mathbf{x}]_{\mathcal{C}}$.

(h) A stochastic matrix that does not have a unique steady-state vector.

2. Let $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - 3z \\ -y + z \end{bmatrix}$.

   (a) (2 points) The codomain of T is ____________________.

   (b) (3 points) Give a basis for the kernel of T.

   (c) (3 points) Give a basis for the range of T.

   (d) (2 points) Is T one-to-one? Explain how you know.

3. (5 points) Suppose B is a 3×4 matrix such that $\text{col}(B) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. What is nullity$(B)$? Clearly explain your reasoning.

4. (5 points) Use a concept from linear algebra to calculate the area of the triangle whose vertices are $(-2,-2)$, $(0,3)$, and $(4,-1)$.

   [Hint: What concept in linear algebra is related to area?]

5. (5 points) Let A be a 4×4 matrix and define

$$S = \{\mathbf{v} \in \mathbb{R}^4 \mid A\mathbf{v} = A^2\mathbf{v}\}.$$

   Is S a subspace? Justify your answer.

6. Suppose $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 3 & -2 & 8 & 12 \\ 2 & 4 & -6 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (2 points) What is the characteristic polynomial of A?

(b) (3 points) $A \begin{bmatrix} 10 \\ 20 \\ 0 \\ 0 \\ 10 \end{bmatrix} =$

(c) (3 points) Is A invertible? Explain how you know.

(d) (3 points) What is $\text{rank}(A + 2I_5)$? Explain.

(e) (4 points) Find all vectors $\mathbf{x}$ in $\mathbb{R}^5$ such that $A\mathbf{x} = \mathbf{x}$. Justify.

7. Consider the following 'Theorem' and its 'Proof'.

Theorem: Let A and B be 2×2 invertible matrices such that $\det A \neq \det B$. Then $A - B$ is invertible as well.

Proof: We know that A is invertible if and only if $\det A \neq 0$. It is therefore enough to show that $\det(A - B) \neq 0$. So, we compute:

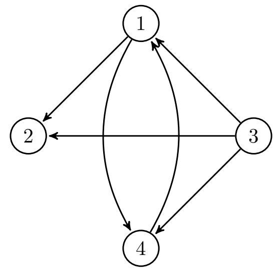
$$\det(A - B) = \det(A) - \det(B) \neq 0$$

where we know $\det(A) - \det(B) \neq 0$ since $\det(A) \neq \det B$. Hence $A - B$ is invertible.

(a) (4 points) This 'Theorem' is not true. Determine the error(s) in the proof. Note that I am **not** asking you to explain why the theorem is false. I am asking you to find flaws in the argument given above.

(b) (4 points) Provide a counterexample that shows that the 'Theorem' is false.

8. Suppose the following graph represents the search results for the query "custom vitamins":



(a) (7 points) Compute the PageRank vector for this search using a damping factor of 0. (Decimals are okay but write your numbers to the .001 place.)

(b) (3 points) Let the damping factor be $p = 1/2$. Provide the matrix M whose eigenspaces you would need to understand to find the PageRank vector (you do not need to actually find the PageRank vector, just the matrix). Do not simplify. Your answer may be given as a linear combination of matrices.

9. Let $A = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 0 & 1 & 0 \\ -4/3 & 2/3 & 1/3 \end{bmatrix}$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Geometrically, T is reflection across a certain plane P in $\mathbb{R}^3$.

   (a) (5 points) Since we know T is reflection across P, what does this tell you about the eigenvalues of A and their multiplicities? Justify geometrically.

   (b) (5 points) Find a basis for the plane P.