Final

for Math 308, Winter 2018

NAME (last - first): ____

- Do not open this exam until you are told to begin. You will have 110 minutes for the exam.
- This exam contains 7 questions for a total of 100 points in 15 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

Question	Points	Score
1	10	
2	8	
3	20	
4	14	
5	18	
6	20	
7	10	
Total:	100	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____ Date: ____________________

Question 1. (10 points) Decide whether the following statements are true or false. For this you don't need to show any work.

(a) [1 point] If $\mathcal{B}$ is a basis of a subspace S and $u \in \mathcal{B}$ then $2u \in \mathcal{B}$.

◯ True ◯ False

(b) [1 point] If a $n \times n$ matrix A is diagonalizable then A is invertible.

◯ True ◯ False

(c) [1 point] If A is a stochastic matrix, $\lambda = 1$ is the smallest eigenvalue of A.

◯ True ◯ False

(d) [1 point] In a subspace of dimension n there are at most n linearly independent vectors.

◯ True ◯ False

(e) [1 point] If u and v are two eigenvectors of a matrix A then u and v are linearly independent.

◯ True ◯ False

(f) [1 point] If a subspace S has a basis $\mathcal{B}$ with 2 elements then S is a plane.

◯ True ◯ False

(g) [1 point] The solution set of a linear system is a subspace.

◯ True ◯ False

(h) [1 point] If A is a $m \times n$ matrix, $\text{rank}(A) \leq \min\{m, n\}$.

◯ True ◯ False

(i) [1 point] If a matrix is diagonalizable, all its eigenvalues are distinct.

◯ True ◯ False

(j) [1 point] If v and w are eigenvectors of A with the same eigenvalue λ then $v - w$ is an eigenvector of A.

◯ True ◯ False

Question 2. (8 points) For any of the following question, give an explicit example. If it is not possible write DNE and explain why.

(a) [1 point] A 3×3 matrix with no more than 2 entries equal to 0 and det $= 0$.

(b) [1 point] A 2×2 matrix with $e_1 + e_2$ as an eigenvector.

(c) [2 point] A 3×3 stochastic matrix with all positive entries and with an eigenvalue $=2$.

(d) [1 point] A subspace S of $\mathbb{R}^4$ of dimension 2 containing e_1 and $3\pi e_2$.

(e) [2 point] Two linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_1 \circ T_2$ is the identity linear transformation, i.e. the linear transformation that sends every x to itself.

(f) [1 point] A 4×4 diagonalizable matrix.

(g) [1 point] A 3×3 matrix with rank $= 2$.

(h) [1 point] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which is onto.

Question 3. (20 points) Let A be the following matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 \\ 0 & 2 & 0 & -1 \end{pmatrix}$$

(note that A is NOT triangular).

(a) [5 points] Compute the characteristic polynomial $p_A(\lambda)$ of A, the eigenvalues of A and their multiplicities.

(b) [2 points] Given your previous computation, is A invertible? Why or why not?

(c) [9 points] Determine a basis for every eigenspace of A.

(d) [4 points] Is A diagonalizable? Why or why not? If it is diagonalizable, find P and D such that $A = P \cdot D \cdot P^{-1}$.

Question 4. (14 points) Let A and T_B be the following matrix and linear transformation (where $a \in \mathbb{R}$):

$$A = \begin{pmatrix} 2 & 1 & 3 & a \\ 0 & 2 & a+1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \qquad\qquad T_B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ x_2 \\ x_2 \end{pmatrix}$$

(a) [3 points] Compute the matrix B associated to the linear transformation T_B, i.e. such that $T_B(x) = B \cdot x$ and identify domain and codomain of both T_A, the linear transformation associated to A, and T_B.

(b) [4 points] For every composition that makes sense, compute the matrices associated to $T_A \circ T_B$ and $T_B \circ T_A$ and identify domain and codomain.

(c) [4 points] Let $C = A \cdot B$. Compute for which values of a, C has rank 2. (Hint: when computing the REF start by switching two rows)

(d) [3 points] For the values of a you found above, compute basis for $\text{row}(C)$ and $\text{col}(C)$ and $\text{null}(C)$.

Question 5. (18 points) Since today is π day you are ask to solve the following linear system in which the coefficients, left to right, top to bottom, are the first digits of the decimal expansion of π.

$$\begin{aligned} 3x_1 + x_2 + 4x_3 + x_4 &= 5 \\ 9x_1 + 2x_2 + 6x_3 + 5x_4 &= 3 \\ 5x_1 + 8x_2 + 9x_3 + 7x_4 &= 9 \end{aligned}$$

(a) [2 points] Where will the solution set live? Will it be a subspace, why?

(b) [3 points] Let A be the augmented matrix of the system. The REF of A is

$$\text{REF of} A = \begin{pmatrix} 1 & 0 & 0 & 0.66 & -0.92 \\ 0 & 1 & 0 & 1 & -0.67 \\ 0 & 0 & 1 & -0.5 & 2.1 \end{pmatrix}$$

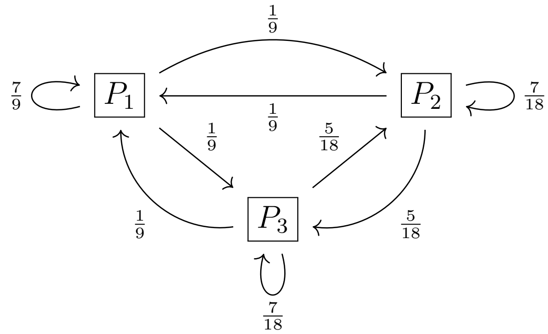
Write down the solution set and compute its dimension.

(c) [2 points] Describe geometrically the solution set.

(d) [7 points] Compute rank(A), nullity(A), basis for col(A), row(A) and null(A). (Hint: you should need very few extra computations).

(e) [4 points] Find 2 vectors $u_1, u_2 \in \mathbb{R}^5$ to obtain a basis of $\mathbb{R}^5$ using the vectors in the basis of row(A) you computed above.

Question 6. (20 points) Consider the following model of the WorldWideWeb consisting of three pages with links.



(a) [4 points] What is the adjacency matrix A of the graph? Explain whether or not it is a Stochastic Matrix.

(b) [3 points] The characteristic polynomial of the matrix A in the previous part is

$$p_A(\lambda) = (\lambda - 1)(\lambda - \frac{2}{3})(\lambda - \frac{1}{3})$$

What are the eigenvalues of A and their multiplicities? Explain why A is diagonalizable.

(c) [2 points] The basis of each eigenspace is given by

$$\mathcal{B}_{E_1(A)} = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \} \quad \mathcal{B}_{E_{2/3}(A)} = \{ \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix} \} \quad \mathcal{B}_{E_{1/3}(A)} = \{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}$$

Compute matrices P and D such that $A = P \cdot D \cdot P^{-1}$. (Choose the following order of the eigenvalues: $1, \frac{2}{3}, \frac{1}{3}$.)

Consider the probability vector

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where each component expresses the probability to start at each page. So in particular we start with probability 100% in page P_1. We want to compute the probabilities to end up in each of the page by computing $A^{20}u$.

(d) [3 points] Compute the steady-state vector v^*, i.e. the equilibrium vector that has the sum of the components equal to 1.

(e) [4 points] Compute the matrix A^{20} using that

$$P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

You can approximate every fraction with a real number with precision 0.001 (3 digits after the .).

(f) [4 points] Compute the vector $A^{20}u$ for the given vector u above and compute the difference $A^{20}u - v^*$ were v^* is the steady-state vector you computed in the previous part.

Extra Credit Question - 10 points

Question 7. (10 points) Suppose a matrix A has the following eigenvectors

$$u_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

with corresponding eigenvalues

$$\lambda_1 = 1 \qquad \lambda_2 = -1 \qquad \lambda_3 = \frac{1}{2}$$

(a) [1 points] Is $\mathcal{B} = \{u_1, u_2, u_3\}$ a basis of $\mathbb{R}^3$? Why?

(b) [2 points] Show that $\operatorname{rank}(A - I_3) = \operatorname{rank}(A + I_3)$.

(c) [2 points] Let $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Express v as a linear combination of the vectors u_1, u_2 and u_3. Use this to compute $A^{2017} \cdot v$.

(d) [5 points] Let $v_k = A^k \cdot v$ for any natural number k. Does the limit $\lim_{k\to\infty} v_k$ exists? If so what is it?