Your Name

Your Signature

- Raise your hand if you have a question or if you need additional paper.
- Turn off and put away all cell phones, laptops, smart watches, and similar devices.
- You may use one 8.5″ × 11″ sheet of *handwritten* notes (both sides OK) and a TI-30X IIS scientific calculator. No other supplementary materials are allowed.
- **You must justify all of your answers.** In order to receive credit, you must show all of your work. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place a box around your answer to each question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.
- I strongly recommend briefly reading through the exam before you begin to answer any questions.

Question	Points	Score
1	16	
2	8	
3	8	
4	12	
5	10	

Question	Points	Score
6	8	
7	8	
8	12	
9	8	
10	10	
Total	100	

Good luck!

1. (16 points) For each of the situations below, either **give an example** (if possible) or **explain why it is not possible**. (If you give an example, you do not need to explain your answer.)

   (a) A nonzero and non-identity matrix A where $A^2 = A$.

   (b) A set of nonzero orthogonal vectors in $\mathbb{R}^3$.

   (c) A diagonalizable matrix that is not invertible.

   (d) An invertible matrix that is not diagonalizable.

(e) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\3\end{bmatrix}, \qquad \text{and} \qquad T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}.$$

(f) A matrix A that is invertible but its inverse A^{-1} is not invertible.

(g) A set of linearly independent vectors that is not orthogonal.

(h) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that is one-to-one.

2. (8 points) Let $\chi_M(\lambda) = (\lambda+1)(\lambda-3)(\lambda+3)(\lambda+2)$.
   (a) M is a ______ $\times$ ______ matrix.

   (b) The eigenvalues of M are ________________________________.

   (c) Is M invertible? Explain.

   (d) Is M guaranteed to be diagonalizable? Explain.

3. (8 points) (a) Find a 3×7 matrix A with the following properties:
   - A is in *reduced* echelon form, and
   - $$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

   (b) The rank of A is ______, and the nullity of A is ______.

4. (12 points) Let $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ -3 & 4 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$. We are told that $\chi_A(A) = (\lambda - 4)^3(\lambda + 2)$.

   (a) For each eigenspace $E_\lambda(A)$, find a basis.

   (b) Find matrices P and D where P is invertible, D is a diagonal matrix, and $A = PDP^{-1}$, *or* explain why it is impossible to do so. (Note: You do not need to calculate P^{-1}; just write down P and D and briefly **explain** where they came from.)

5. (10 points) The linear transformation T is defined

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_3 \\ 2x_2 - x_3 \end{bmatrix}.$$

(a) The domain of T is ____________, and the codomain of T is ____________.

(b) Find a matrix A such that $T(\vec{x}) = A\vec{x}$.

(c) Find its inverse transformation T^{-1} or explain why T^{-1} cannot exist.

(d) Is T one-to-one? Is T onto? Explain your answers.

6. (8 points) Consider the matrix $A = \begin{bmatrix} 2 & 7 \\ 2 & -3 \end{bmatrix}$.

   (a) Calculate $\chi_A(\lambda)$, the characteristic polynomial of A.

   Given a polynomial $f(x)$ and a matrix M, we can evaluate $f(M)$ like in the following example:

$$f(x) = 2x^2 - 5 \qquad M = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \qquad M^2 = \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}$$

$$f(M) = 2(M^2) - 5I = 2\begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ -8 & -1 \end{bmatrix}$$

   (Notice that the constant term c becomes cI, c times the identity matrix.)

   (b) Evaluate $\chi_A(A)$ for the matrix and characteristic polynomial above.

   (c) Make a conjecture for the value of $\chi_M(M)$ for matrices and their characteristic polynomials in general. *You do not need to show any additional work for this part.*

7. (8 points) We are told that A and B are both $n \times n$ matrices and that $\vec{v}$ is an eigenvector for *both* A and B. Let λ_1 be the eigenvalue associated with A and $\vec{v}$. Let λ_2 be the eigenvalue associated with B and $\vec{v}$.

   (a) Show that $\vec{v}$ is an eigenvector for the matrix AB.

   (b) What is the associated eigenvalue for AB and $\vec{v}$?

   (c) Assume that A is invertible. Show that $\vec{v}$ is an eigenvector for A^{-1}.

   (d) What is the associated eigenvalue for A^{-1} and $\vec{v}$?

8. (12 points) Let S be the subspace defined

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(a) Find a basis for $S^\perp$. ($S^\perp$ is the set of all vectors $\vec{v}$ in $\mathbb{R}^4$ where $\vec{v} \cdot \vec{s} = 0$ for all $\vec{s} \in S$.)

(b) Find an orthogonal basis for S.

9. (8 points) S is a subspace of $\mathbb{R}^3$. The following sets β_1 and β_2 are both bases for S.

$$\beta_1 = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \beta_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$$

(a) If $[\vec{x}]_{\beta_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, find $[\vec{x}]_{\beta_2}$.

(b) Find $\vec{x}$, the representation of this vector in the standard basis.

10. (10 points) Let $A = \begin{bmatrix} x & -5 & -1 \\ 2 & 2 & x \\ 0 & -1 & -1 \end{bmatrix}$.

   (a) Find the value(s) of x for which the matrix A is *not* invertible.

   (b) Pick one of the value(s) of x from part (a) and use it for this part. For this value of x, find a basis for col(A). (If you cannot solve (a), pick any value for x for this part.)

   I have picked the value $x =$ ______.