(1) Find the determinant of the matrix
\[
\begin{bmatrix}
1 & 3 & 2 \\
0 & 1 & 1 \\
-2 & 0 & 4 \\
\end{bmatrix}
\begin{bmatrix}
8 & 0 & 3 \\
-1 & 1 & 1 \\
0 & 2 & 4 \\
\end{bmatrix}^{-1}.
\]

(2) (Geometry Question) (Note: This problem is repeated in Chapter 3 with fewer parts.) Suppose we are given the unit square \( A \) in the plane with corners \((0,0), (1,0), (1,1)\) and \((0,1)\).

(a) Find a linear transformation \( T \) that sends \( A \) to the parallelogram \( B \) with corners \((0,0), (1,2), (2,2)\) and \((1,0)\).

(b) Where does \( T \) send the point \((1/2,1/2)\), which was in \( A \)?

(c) Is the linear transformation \( T \) unique? Why or why not?

(d) What linear transformation \( T' \) would send \( A \) to itself?

(e) Calculate the area of \( B \). Do you see a relationship between this area and the matrix of the linear transformation \( T \)? Similarly is there a relationship between the area of \( A \) and \( T' \)?

(f) Suppose we want to not only send \( A \) to \( B \) but also push \( B \) in the horizontal direction by one unit. What map can do this?

(g) Let \( L \) be the linear span of the side of \( B \) with corners \((0,0)\) and \((1,2)\). Write \( L \) in parametric form: \( \mathbf{p} + t\mathbf{q} \) where \( t \) varies in some range and \( \mathbf{p}, \mathbf{q} \) are vectors. What is the range of \( t \) and what are \( \mathbf{p} \) and \( \mathbf{q} \)?

(h) Find the point in \( A \) that maps under \( T \) to the point \((1/2,1)\) on \( L \). In your parametric representation of \( L \), what is the representation of \((1/2,1)\)?

(i) How can you map \( A \) to a parallelogram \( C \) of area 4 while still keeping \((0,0)\) and \((1,0)\) as two of its corners?

(j) What is the general formula for the linear transformation that sends \( A \) to a parallelogram of area \( k \) while still keeping \((0,0)\) and \((1,0)\) as two of its corners?

(3) (The math world’s worst formula for computing inverses)
Let \( A = \begin{bmatrix} -2 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & -1 & 5 \end{bmatrix} \).

(a) Compute all nine cofactors of \( A \), as well as \( \det(A) \). Let \( B \) be the \( 3 \times 3 \) matrix containing the cofactors, with each entry multiplied by the appropriate \( \pm \) sign. So the \( ij \)-entry of \( B \) is \((-1)^{i+j} \det(M_{ij})\).

(b) Compute \( A \cdot B^T \). You should get a diagonal matrix with the same number in every diagonal entry. In other words, a multiple of the identity matrix. What multiple is it (in terms of \( A \))? 

(c) Fill in the blank (with a scalar) to make this equation true:
\[
A \cdot B^T = (\ ? ) \cdot I, \quad \text{therefore} \quad A^{-1} = \frac{1}{(\ ? )} \cdot B^T.
\]
(d) A similar formula works for larger $n \times n$ matrices, involving computing all the cofactors of $A$. But this formula is terrible for computational purposes for finding $A^{-1}$. Why? Compare it to our other method. (Note: Occasionally the formula is useful for theoretical purposes.)

(4) (Determinants and geometry)

(a) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation by $\pi/3$, i.e. $T(\vec{x})$ is the rotation of $\vec{x}$ by $\pi/3$ around $\vec{0}$. Without computing any matrices, what would you expect $\det(T)$ to be? (Does $T$ make areas larger or smaller?)

Guess, then check using the fact that the matrix for rotation by $\theta$ is

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$ 

(b) Same question as (a), only this time let $T$ be the transformation that reflects $\mathbb{R}^2$ over the line $y = x$. That is, $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ x \end{bmatrix}$. Guess what $\det(T)$ should be, then check by finding the matrix for $T$ and computing its determinant.

(c) Rotation matrices in $\mathbb{R}^3$ are more complicated than in $\mathbb{R}^2$ because you have to specify an axis of rotation, which could be any line through the origin. Nonetheless, what would you expect $\det(T)$ to be? Look up the “basic 3D rotation matrices” on Wikipedia (https://en.wikipedia.org/wiki/Rotation_matrix#In_three_dimensions) and compute $\det(A)$ for each one.

(d) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be projection onto the $xy$-plane, so $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. What is $\det(T)$? Guess, then check using a matrix.

(5) (Determinants and interpolation)

Suppose we want to make a quadratic polynomial

$$y = f(x) = a_0 + a_1 x + a_2 x^2$$

that passes through three specified points $\mathbf{p}_1 = (p_1, q_1), \mathbf{p}_2 = (p_2, q_2), \mathbf{p}_3 = (p_3, q_3)$. Consider the equation

$$0 = \det \begin{bmatrix} 1 & x & x^2 & y \\ 1 & p_1 & p_1^2 & q_1 \\ 1 & p_2 & p_2^2 & q_2 \\ 1 & p_3 & p_3^2 & q_3 \end{bmatrix}.$$ 

The determinant above implicitly gives an equation $y = f(x)$ (it’s easy to solve for $y$ since no $y^2, y^3$, etc terms appear).

(a) Write out the matrix above, using $\mathbf{p}_1 = (0, 0), \mathbf{p}_2 = (1, 1), \mathbf{p}_3 = (3, 5)$ for the constants $p_i, q_i$, but leaving $x$ and $y$ as variables. Solve the equation $\det(A) = 0$ to get $y = f(x)$, a quadratic polynomial in $x$. Check directly that the parabola passes through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. 
(b) Why does part (a) succeed? Examine the matrix $A$ from part (a). If you plug in $(x, y) = p_1 = (0, 0)$ to the first row of $A$, the first two rows will become the same. So, by the ‘repeated rows’ rule, the equation $\det(A) = 0$ must be true for those specific $x, y$ values. What does this mean about the polynomial $y = f(x)$?

What about if you plug in $(x, y) = (1, 1)$ or $(3, 5)$? Why (in terms of determinants) must the equation $y = f(x)$ be satisfied by these points?

(c) Try to generalize: how could you use a determinant to make a cubic polynomial that passes through 4 given points? (It should require a $5 \times 5$ determinant.)

(6) Find a formula of the form $x = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ for a matrix solution of the quadratic equation $ax^2 + bx + c = 0$. Here $c$ denotes $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ and 0 denotes $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(Hint: First show how the square root of any number $D$ can be obtained using a matrix of the form $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, where it looks different depending on whether $D$ is positive or negative. Then use the quadratic formula.)

(7) Let $A(t) = b_1 \cos(\omega t) + b_2 \sin(\omega t)$ be the ambient temperature, which varies sinusoidally. We suppose that $A(t)$ is known — that is, the values of $b_1, b_2,$ and $\omega$ have been measured. Newton’s Law of Cooling states that the temperature function $y(t)$ satisfies

$$y' = -k(y - A(t))$$

with $k$ a (known) positive constant. It turns out that the “steady-state” solution (which $y$ always approaches in the limit as $t$ increases) is of the form

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Use Newton’s Law of Cooling to find linear relations between the $b_i$ and the $c_i$. Write the linear relations as a system of equations with $kb_i$ on the right side. Then use an inverse matrix to find a formula for the steady-state solution.

(8) Let

$$f(t) = x_1 \sin(t) + x_2 \cos(t) + x_3 t \sin(t) + x_4 t \cos(t)$$

and

$$f'(t) = y_1 \sin(t) + y_2 \cos(t) + y_3 t \sin(t) + y_4 t \cos(t),$$

where the $x_i$ and $y_i$ are coefficients. Find a matrix $A$ expressing $\vec{y}$ as a linear transformation $T(\vec{x})$. Also answer the question: What is the significance of $A^2$ and $A^3$?