**Spring 2018 MATH 307 Midterm 2**
**80 pts total**

**Name:** Heather

Instruction:

- Nothing but writing utensils and a double side 4in $\times$ 6in notecard are allowed.
- Unless otherwise specified, you must show work to receive full credit.

1 (24pts). Find the solution to the initial value problem

$$y'' - y = xe^x, \qquad y(0) = 1, \quad y'(0) = -1/4$$

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Char eq: $r^2 - 1 = 0$

$\Rightarrow (r+1)(r-1) = 0$

$\Rightarrow r = 1, -1$

so $y = c_1 e^x + c_2 e^{-x} + y_p(x)$

Ansatz: $y_p(x) = x(Ax+B)e^x = Ax^2e^x + Bxe^x$

$y_p'' - y_p = xe^x$

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$y_p' = Ax^2e^x + 2Axe^x + Bxe^x + Be^x$

$= Ax^2e^x + (2A+B)xe^x + Be^x$

$y_p'' = Ax^2e^x + 2Axe^x + (2A+B)xe^x + (2A+B)e^x + Be^x$

$xe^x = y_p'' - y_p = Ax^2e^x + (4A+B)xe^x + (2A+2B)e^x - (Ax^2e^x + Bxe^x)$

$= 4Axe^x + (2A+2B)e^x$

$\Rightarrow A = \frac{1}{4}, \quad B = -A = -\frac{1}{4}$

$\Rightarrow \boxed{y = c_1e^x + c_2e^{-x} + \frac{1}{4}x^2e^x - \frac{1}{4}xe^x}$

$1 = y(0) = c_1 + c_2$

$y'(x) = c_1e^x - c_2e^{-x} + \frac{1}{2}xe^x + \frac{1}{4}x^2e^x - \frac{1}{4}xe^x - \frac{1}{4}e^x$

$-\frac{1}{4} = y'(0) = c_1 - c_2 - \frac{1}{4}$

$0 = c_1 - c_2$

$\left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = \frac{1}{2}$, $\boxed{y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} + \frac{1}{4}x^2e^x - \frac{1}{4}xe^x}$

*(Extra space just in case you need it.)*

2 (12pts). Consider the linear homogeneous equation

$$t^2y'' - 4ty' + 6y = 0, \quad t > 0.$$

[a] Find all values of $p$ such that $y(t) = t^p$ is a solution to the above equation.
[b] Find the general solution to the differential equation.

(a) $y(t) = t^p$ is a soln if (note: $y' = pt^{p-1}$, $y'' = p(p-1)t^{p-2}$)

$$t^2 p(p-1)t^{p-2} - 4tpt^{p-1} + 6t^p = 0$$

$$(p^2-p)t^p - 4pt^p + 6t^p = 0$$

$$(p^2-p-4p+6)t^p = 0$$

$$p^2 - 5p + 6 = 0$$

$$(p-2)(p-3) = 0$$

$$\Rightarrow \boxed{p = 2, \quad p = 3}$$

(b) $y_1(t) = t^2$, $y_2(t) = t^3$ are two independent soln to this homogeneous linear 2nd order ODE.

So $\boxed{y(t) = c_1 t^2 + c_2 t^3}$ is the general soln.

3. Hanging a mass of $m = 1\ kg$ stretches the spring by $5/(2\pi^2)$ meters. Use $g = 10\ m/s^2$ as the acceleration due to gravity.

[a] (4pts) Find the spring constant $k$.

$$k\left(\frac{5}{2\pi^2}\right) = mg = 10 \quad \Rightarrow \quad k = \frac{10 \cdot 2\pi^2}{5} = \boxed{4\pi^2}$$

[b] (4pts) Denote by $y(t)$ the displacement the mass at any time $t$, with $y(t) > 0$ when the spring is stretched from the equilibrium position and $y(t) < 0$ when the spring is compressed. Write down the differential equation that governs the motion of this undamped mass-spring system. *Note: you need to put in actual numbers, not just symbols, for the mass, spring constant, and etc., to get full credit. No need to show work.*

$$y'' + 4\pi^2 y = 0$$

[c] (16pts) Suppose the initial displacement is $-\frac{1}{4\sqrt{2}}$ and the initial velocity is $\frac{\pi}{2\sqrt{2}}$. Find $y(t)$ for all $t$ and express the answer in the form of $y(t) = A\cos(\omega t - \phi)$.
*(I left more space for you to write on the next page in case you need that.)*

$$r^2 + 4\pi^2 = 0 \quad \Rightarrow \quad r = \pm 2\pi i$$

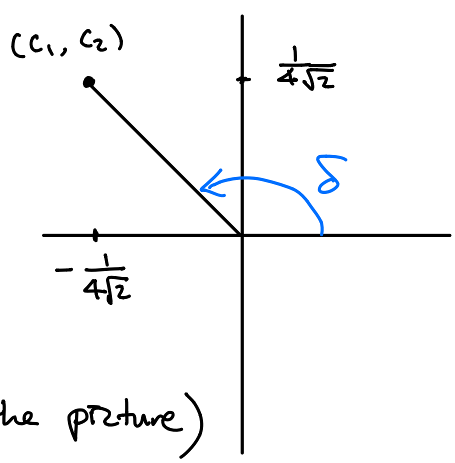
$$y(t) = c_1 \cos(2\pi t) + c_2 \sin(2\pi t)$$

$$-\frac{1}{4\sqrt{2}} = y(0) = c_1 \quad \Rightarrow \quad c_1 = -\frac{1}{4\sqrt{2}}$$

$$y'(t) = -2\pi c_1 \sin(2\pi t) + 2\pi c_2 \cos(2\pi t)$$

$$\frac{\pi}{2\sqrt{2}} = y'(0) = 2\pi c_2 \quad \Rightarrow \quad c_2 = \frac{1}{4\sqrt{2}}$$

$$y = -\frac{1}{4\sqrt{2}}\cos(2\pi t) + \frac{1}{4\sqrt{2}}\sin(2\pi t) = A\cos(2\pi - \delta)$$




$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{32} + \frac{1}{32}} = \frac{1}{4}$$

$$\tan\delta = \frac{c_2}{c_1} = -1$$

$$\delta = \tan^{-1}(-1) + \pi$$

$\delta = \frac{3\pi}{4}$ (can also see from the picture)

$$\Rightarrow \boxed{y = \frac{1}{4}\cos\left(2\pi t - \frac{3\pi}{4}\right)}$$

3. (Continued)

Amplitude = 0.25

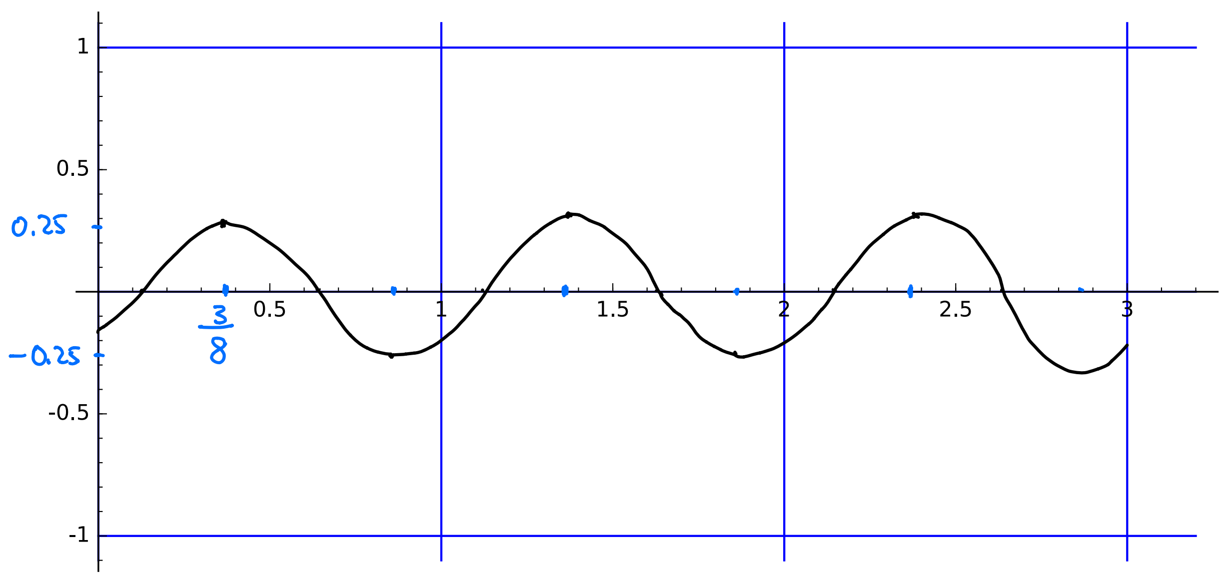
period $= \frac{2\pi}{2\pi} = 1$

First peak of cos happens when

$$2\pi t - \frac{3\pi}{4} = 0$$

$$t = \frac{3\pi}{4(2\pi)} = \frac{3}{8} = 0.375$$

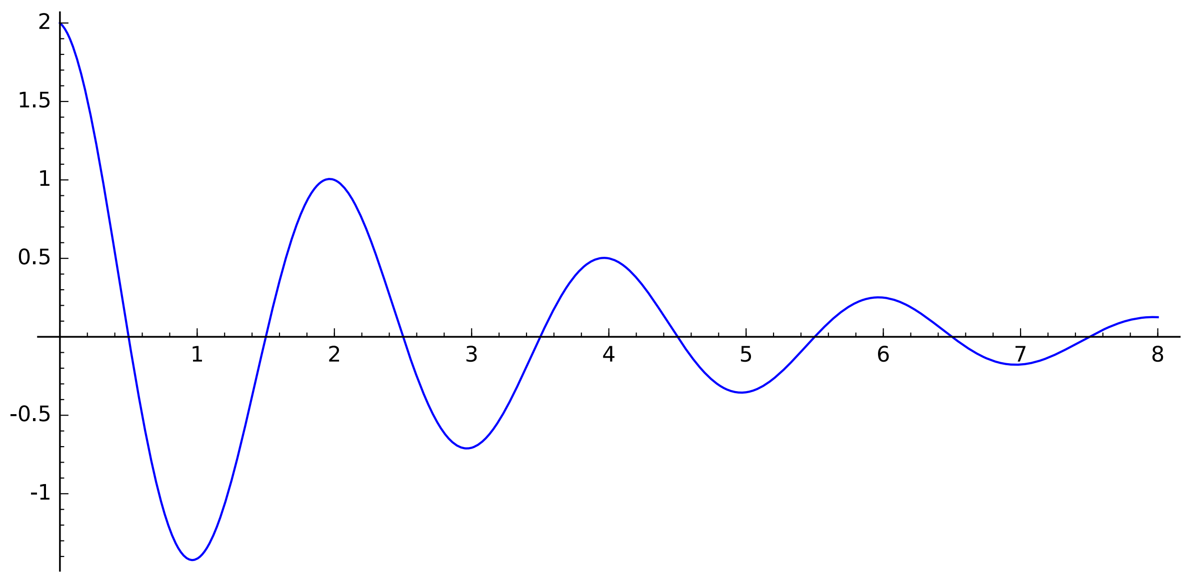
[d] (6pts) Graph the solution (be sure your graph illustrates the period, amplitude, and phase shift accurately). *You don't have to explain.*



4 (14pts). You have a spring in a damping media, but you don't know the spring constant nor the damping constant. To find that out, you decide to attach a mass of 1 *kg* to the spring and plot the motion of this unforced damped spring-mass system. The graph below is a plot of the displacement of the mass at any time $t$. Write down the differential equation governing its motion.

*Explain your reasoning to get full credit. Note: you should write down actual (estimated) numbers based on what you gather from the graph, not just a symbolic equation.*

*The next page is blank in case you need more space to work.*




free vibration w/ damping:

$m = 1$

$$y'' + \gamma y' + ky = 0$$

char eq: $-\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 4k}}{2} = -\frac{\gamma}{2} \pm i \underbrace{\frac{\sqrt{4k - \gamma^2}}{2}}_{= \mu}$

Soln: $y(t) = C_1 e^{-\frac{\gamma}{2}t} \cos \mu t + C_2 e^{-\frac{\gamma}{2}} \sin \mu t$

$$= A e^{-\frac{\gamma}{2}t} \cos(\mu t - \delta)$$

Can immediately read off from the picture that

$A = 2$, $\delta = 0$, (quasi)-period $= 2$ $\Rightarrow$ $\mu = \pi$

$\Rightarrow$ continue next page

*(Extra space in case you need it.)*

So, $y(t) = 2e^{-\frac{\gamma}{2}t}\cos(\pi t)$

To figure out $\gamma$:

$$\underbrace{1 = y(2)}_{\text{read off the graph}} = 2e^{-\frac{\gamma}{2}(2)}\cos(2\pi) = 2e^{-\gamma}$$

$$e^{-\gamma} = \frac{1}{2}$$

$$-\gamma = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

$$\gamma = \ln(2) \approx 0.69$$

To figure out $k$:

$$\pi = \mu = \frac{\sqrt{4k - \gamma^2}}{2}$$

$$\Rightarrow 4\pi^2 = 4k - (\ln(2))^2$$

$$\Rightarrow \quad k = \pi^2 + \left(\frac{\ln(2)}{2}\right)^2 \approx 9.99$$

$$\boxed{y'' + \ln(2)\,y' + \left(\pi^2 + \left(\frac{\ln(2)}{2}\right)^2\right)y = 0}$$